4 Differentiation and integration

Differentiation enables us to calculate rates of change. This is very useful for finding expressions for displacement, velocity and acceleration. For example, an expression for the displacement of an airplane in the sky informs us of the distance and direction of the airplane from its original position, at a given time. The first derivative of this expression gives the rate of change of displacement, that is, the velocity. The second derivative gives an expression for the rate of change of velocity, that is, the acceleration.

Differentiation and integration (which can be considered as reverse differentiation) belong to a branch of mathematics called calculus. Calculus is the study of change, and it's a powerful tool in modelling real-world situations. Calculus has many applications in a variety of fields including quantum mechanics, thermodynamics, engineering and economics, in modelling growth and movement.

Orientation

What you need to know

p.26

Ch1.6 Lines and circles

- The equation of a straight line where m is the gradient.
- When two lines are perpendicular to each other: $m_1 \times m_2 = -1$

p.43

Ch2 Polynomials and the binomial theorem

- Binomial expansion formula.
- Curve sketching.

What you will learn

- To differentiate from first principles.
- To differentiate terms of the form ax^n
- To calculate rates of change.
- To work out and interpret equations, tangents, normals, turning points and second derivatives.
- To work out the integral of a function, calculate definite integrals and use these to calculate the area under a curve.

What this leads to

Ch7 Units and kinematics
Velocity and acceleration as rates of change.
Acceleration as a second derivative.
Area under a velocity-time graph.

Ch15 Differentiation
Points of inflection.
The product and quotient rules.
The chain rule.

Ch16 Integration
Integration by parts, by substitution and using partial fractions.

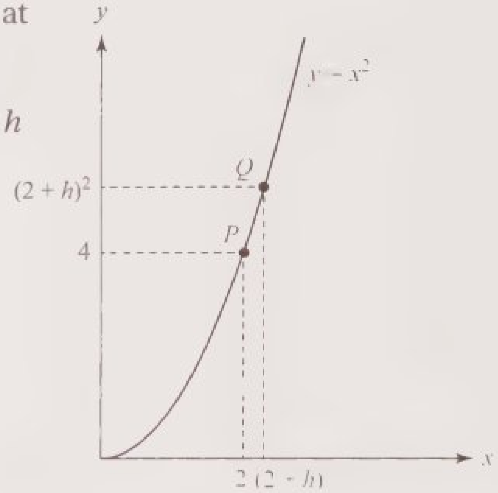
MyMaths Practise before you start — 2002, 2004, 2022, 2041

4.1 Differentiation from first principles

Fluency and skills

When looking at the graph of a function, the gradient of its curve at any given point tells you the rate of change. Differentiation from first principles is a method of calculating the gradient and, therefore, the rate of change.

For example, you can work out the gradient of the function $y = x^2$ at the point $P(2, 4)$ using differentiation from first principles.



Define a point Q that lies on the curve, a tiny horizontal distance h from P, so that Q has coordinates $(2 + h, (2 + h)^2)$

PQ is the chord that connects the points.

The gradient of the chord PQ is given by

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{(2+h)^2 - 4}{(2+h) - 2}$$

$$= \frac{4 + 4h + h^2 - 4}{(2+h) - 2} = \frac{4h + h^2}{h}$$

$$= 4 + h$$

As the distance between P and Q becomes very small, h very small and m_{PQ} approaches 4

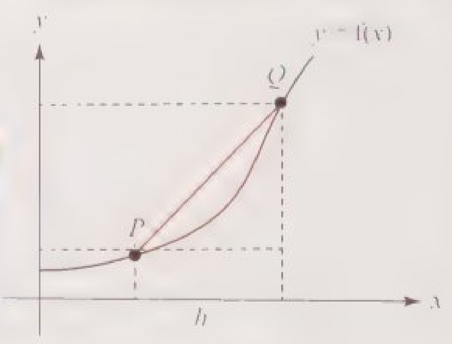
Gradient at $P = 4$

This method can be generalised for any function.

ICT Resource online

To investigate gradients of chords for a graph, click this link in the digital book.

Consider the graph $y = \mathrm{f}(x)$
Let the point P lie on the curve and have x-coordinate x
Its y-coordinate is then $\mathrm{f}(x)$
Let the point Q also lie on the curve, h units to the right of P
Its coordinates are therefore $(x + h, \mathrm{f}(x + h))$



Key point

The **gradient** of PQ is given by

$$m_{PQ} = \frac{y_Q - y_P}{x_Q - x_P} = \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{(x+h) - x} = \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}$$

As h approaches 0, the point Q approaches P and the gradient of the chord PQ gets closer to the gradient of the curve at P

The gradient of the curve at P is defined as the **limiting value** of the gradient of PQ as h approaches 0. This limit is denoted by $f'(x)$ and is called the **derived function** or **derivative** of $f(x)$. See p.300 for a list of mathematical notation.

Key point

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

A limiting value, or **limit**, is a specific value that a function approaches or tends towards. "$\lim_{h\to 0}$" followed by a function means the limit of the function as h tends to zero.

Differentiation with this method is referred to as finding the derivative from **first principles**.

Example 1

Use differentiation from first principles to work out the derivative of $y = x^2$ and the gradient at the point (3, 9)

$f(x) = x^2$

So, $f(x+h) = (x+h)^2$

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$ — Substitute the function $f(x) = x^2$

$= \lim_{h\to 0}(2x+h)$ — Expand and simplify the expression.

$= 2x$ — Let h tend towards zero.

The gradient at the point (3, 9) is $f'(3) = 2\times 3 = 6$

Derivatives give the rate of change and **constants** don't change. So a function multiplied by a constant will differentiate to give the derived function multiplied by the *same constant*.

If a function is itself a constant, then the derivative will be zero.

Key point

For a function $af(x)$, where a is a constant, the derived function is given by $af'(x)$

For a function $f(x) = a$, where a is a constant, the derived function $f'(x)$ is zero

Example 2

Differentiate **a** $3x^2$ **b** 7

a $f(x) = x^2$, $a = 3$ — $3x^2$ is in the form $af(x)$, where a is a constant.

So differentiating $3x^2$:

$3\times 2x = 6x$ — From Example 1, $f'(x) = 2x$

b $f(x) = 7$ — $f(x) = a$, where a is a constant.

So $f'(x) = 0$

Exercise 4.1A Fluency and skills

1 Use the method shown in Example 1 to work out the gradient of these functions at the points given.

a $y=x^2$ at $x=3$ b $y=x^2$ at $x=4$

c $y=2x^2$ at $x=2$ d $y=5x^2$ at $x=1$

e $y=\frac{1}{2}x^2$ at $x=4$ f $y=\frac{3}{4}x^2$ at $x=10$

g $y=x^3$ at $x=1$ h $y=3x^3$ at $x=2$

2 Use the method of differentiation from first principles to work out the derivative and hence the gradient of the curve.

a $y=x^2$ at the point $(1,1)$

b $y=3x^2$ at the point $(2,12)$

c $y=x^3$ when $x=3$

d $y=4x-1$ at the point $(5,19)$

e $y=\frac{1}{2}x^2$ at the point $(6,18)$

f $y=x^2+1$ at the point $(2,5)$

g $y=2x^3$ at the point $(1,2)$

h $y=x^3+2$ at the point $(2,10)$

3 Use differentiation from first principles to work out the gradient of the tangent to

a $y=2x^3$ at the point where $x=2$

b $y=3x^2+2$ at $(3,29)$

c $y=\frac{x^2}{2}$ at the point $(4,8)$

d $y=\frac{1}{2}x^2-4$ at the point $(2,-2)$

e $y=2x^2-1$ at the point $(-2,7)$

f $y=x^2+x$ at the point where $x=1$

g $y=x^3+x$ at the point $(2,10)$

4 Work out, from first principles, the derived function when

a $f(x)=2x^2$ b $f(x)=4x^2$

c $f(x)=6x^2$ d $f(x)=\frac{1}{2}x^2$

e $f(x)=x^2+1$ f $f(x)=2x-3$

g $f(x)=\frac{1}{3}x^3$ h $f(x)=2x^3+1$

5 a Work out, from first principles, the derived function where

i $f(x)=x+x^2$

ii $f(x)=x^2+x+1$

iii $f(x)=x^2+x-5$

iv $f(x)=x^2+2x+3$

v $f(x)=x^2-3x-1$

vi $f(x)=2x^2+5x-3$

b Work out, from first principles, the derived function of

i $f(x)=6$ ii $f(x)=0$

iii $f(x)=-2$ iv $f(x)=\pi$

c Work out, from first principles, the derived function of

i $f(x)=x$ ii $f(x)=-x$

iii $f(x)=2x+1$ iv $f(x)=4-3x$

6 Differentiate

a $y=2x^3-3$ b $y=3x^4-2x^2$

c $y=2x^5$ d $y=1-x^3$

e $y=x^2-2x^4$ f $y=x-x^2+x^4$

Reasoning and problem-solving

Strategy

To solve problems involving differentiation from first principles

1. Substitute your function into the formula $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$
2. Expand and simplify the expression.
3. Let h tend towards 0 and write down the limit of the expression, $f'(x)$
4. Find the value of the gradient at a point (a, b) on the curve by evaluating $f'(a)$

Example 3

At which point on the curve $y = 3x^2$ is the gradient equal to 18?

$f(x) = 3x^2$ so, $f'(x) = \lim_{h\to 0}\frac{3(x+h)^2 - 3x^2}{h}$ — 1 Apply the algebra.

$= \lim_{h\to 0}\frac{3x^2 + 6xh + 3h^2 - 3x^2}{h} = \lim_{h\to 0}(6x + 3h)$

$f'(x) = 6x$ — 2 Find the derivative.

$f'(x) = 18$ So, $6x = 18$ $x = 3$ — 3 Use the derivative and the initial conditions to form an equation.

If $x = 3$ then $y = f(3) = 3 \times 3^2 = 27$

The gradient is 18 at the point (3, 27) — 4 Solve the equation you formed.

Exercise 4.1B Reasoning and problem-solving

1 At which point on the curve $y = 5x^2$ does the gradient take the value given? Show your working.

a 20

b 100

c 0.5

d Half of what it is at (2, 20)

e A third of what it is at (3, 45)

f Four times what it is at (1, 5)

2 At which **two** points on the curve $y = x^3$ does the gradient equal the value given? Show your working.

a 3 b 12 c 27

d 0.03 e $\frac{1}{3}$ f 1.47

3 For what value of x will these pairs of curves have the same gradient? Show your working.

a $y = x^2$ and $y = 2x$

b $y = 2x^2$ and $y = 12x$

c $y = 3x^2$ and $y = 15x$

d $y = ax^2$ and $y = bx$ where a and b are constants.

4 For what value(s) of x will these pairs of curves have the same gradient? Show your working.

a $y = x^3$ and $y = x^2 + 5x$

b $y = x^3$ and $y = 3x^2 + 9x$

c $y = x^3$ and $y = 2x^2 - x$

d $y = x^3$ and $y = 4x^2 - 4x$

e $y = x^2 + 3x + 1$ and $y = 7x + 1$

f $y = 2x^2 + x - 4$ and $y = 11x + 2$

g $y = x^2 - x$ and $y = 5x - 2$

h $y = 2x^3 + 3x^2$ and $y = 12x$

5 a Consider the derivatives of 1, x, x^2, x^3, x^4, and hence suggest a general rule about the derivative of x^n

b Consider the derivatives of 2, $4x$, $3x^2$, $5x^3$, $-2x^4$ and hence suggest a general rule about the derivative of ax^n

c Test your rule by considering the derivative of $2x^3$ and $3x^2$

d Can you say you have proved your rule?

Challenge

6 Can you show from first principles that

a The curve $y = \frac{1}{x}$ has a gradient of −4 when $x = \frac{1}{2}$

b The derivative of $\frac{1}{x}$ is $-\frac{1}{x^2}$?

4.2 Differentiating ax^n and Leibniz notation

Fluency and skills

You can use a simple rule to differentiate functions in the form ax^n

Key point

If $f(x) = ax^n$ then $f'(x) = nax^{n-1}$

If a function is a sum of two other functions, you can differentiate each function one at a time and then add the results.

Key point

If $h(x) = f(x) + g(x)$ then $h'(x) = f'(x) + g'(x)$

Isaac Newton is credited with having come up with the idea of Calculus first, but **Gottfried Leibniz**, a German mathematician and contemporary of Newton, also developed the concept and devised an alternative notation which is commonly used.

See p.300 For a list of mathematical notation.

For $y_Q - y_P$ he used the symbol δy and for $x_Q - x_P$ he used the symbol δx

So $\frac{y_Q - y_P}{x_Q - x_P} = \frac{\delta y}{\delta x}$ and he wrote that $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

Key point

If $y = f(x)$ then $\frac{dy}{dx} = f'(x)$

Differentiating a function, or finding $\frac{dy}{dx}$, gives a formula for the gradient of the graph of the function at a point. This is also the gradient of the tangent to the curve at this point.

Try it on your calculator

You can use a calculator to evaluate the gradient of the tangent to a curve at a given point.

$\frac{d}{dx}(5X^2 - 2X)\big|_{x=3}$

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Activity

Find out how to calculate the gradient of the tangent to the curve $y = 5x^2 - 2x$ where $x = 3$ on *your* calculator.

Example 1

Differentiate $y = 3x^5 + 4x^2 + 2x + 3$

$y = 3x^5 + 4x^2 + 2x + 3 = 3x^5 + 4x^2 + 2x^1 + 3x^0$ — Write each term in the form ax^n

$\frac{dy}{dx} = 5 \times 3x^4 + 2 \times 4x^1 + 1 \times 2x^0 + 0 \times 3x^{-1}$ — Use $f'(x) = nax^{n-1}$ on each term.

$\frac{dy}{dx} = 15x^4 + 8x + 2$

Example 2

Work out the derived function when $f(x)=4+\frac{3}{x}+\sqrt{x}+2x^7$

$f(x)=4x^0+3x^{-1}+x^{\frac{1}{2}}+2x^7$ — Write each term in the form ax^n

$f'(x)=0\times4x^{-1}+(-1)\times3x^{-2}+\frac{1}{2}\times x^{-\frac{1}{2}}+7\times2x^6$ — Use $f'(x)=nax^{n-1}$ on each term.

$f'(x)=-3x^{-2}+\frac{1}{2}x^{-\frac{1}{2}}+14x^6$

$f'(x)=-\frac{3}{x^2}+\frac{1}{2\sqrt{x}}+14x^6$ — Simplify.

Exercise 4.2A Fluency and skills

1 Differentiate

a $3x^7$ b $6x^4$ c $5x^2$

d 8 e -4 f $\frac{1}{3}$

g $-2x^5$ h $-x^7$ i $2x^{-1}$

j $-x^{-3}$ k $x^{\frac{1}{2}}$ l $x^{\frac{7}{2}}$

m $5x^{\text{[illegible]}}$ n $-x^{\text{[illegible]}}$ o 0

2 Work out the derivative of

a $\sqrt{x}$ b $\sqrt[3]{x}$ c $\sqrt[5]{x}$

d $\sqrt{x^3}$ e $\sqrt[3]{x^2}$ f $\sqrt[3]{3x}$

g $\frac{1}{x}$ h $\frac{3}{x^2}$ i $-\frac{6}{x^4}$

j $\frac{1}{\sqrt{x}}$ k $\frac{3}{\sqrt{x^5}}$ l $\frac{1}{\sqrt{5}}$

m π

3 Calculate $\frac{dy}{dx}$ when $y=$

a x^2+2x-3 b $1-2x-5x^2$

c x^3+2x^2-3x+1 d $x^2-x^1+\pi$

e $x-\frac{1}{x}$ f $3+x+\frac{2}{x^2}$

g $x^3-\frac{1}{x^3}+\frac{3}{x}+5$ h $\frac{10}{x}-1-\frac{x}{10}$

i $\sqrt{x}+\frac{1}{\sqrt{x}}$ j $x^5-\frac{5}{\sqrt[4]{x}}$

k $3+\frac{2}{\sqrt{x}}-\frac{5}{\sqrt[3]{x^2}}$ l $\frac{3}{\pi}-\frac{2}{x^2}-x^{\frac{1}{2}}$

4 For each question part **a** to **d**

i Find an expression in x for the gradient function,

ii Find the value of the gradient at the given point.

a Given that $f(x)=3x^2+4x-6$ work out the value of $f'(-2)$

b Given that $y=2x^3-5x^2-1$ work out the value of $\frac{dy}{dx}$ when $x=1$

c A curve has equation $y=10x+\frac{8}{x}$
Calculate the gradient of the curve at the point where $x=2$

d Calculate the gradient of the curve $y=\frac{8}{x}+\frac{4}{x^2}$ at the point $(2,5)$

e i Expand $y=x(x-1)$

ii Hence evaluate $\frac{dy}{dx}$ when $x=4$ and $y=x(x-1)$

iii Hence state the gradient of the tangent to the curve at $(4,12)$

5 Write an expression in x for $\frac{dy}{dx}$ and thus calculate the gradient of the tangent to each curve at the point given.

a $y=2x^2-5x+1$ at $(1,-2)$

b $y=1-5x+\frac{10}{x}$ at $(2,-4)$

c $y=x(2x+1)$ at $(-3,15)$

d $y=\sqrt{x}+\frac{2}{\sqrt{x}}$ at $(4,3)$

6 For each pair of functions, find which has the greater gradient at the given point.

a $y=x^2$ and $y=20-x$ at the point (4, 16)

b $y=x^2+3x-8$ and $y=6-2x$ at the point (2,2)

c $y=2x^2+13x-18$ and $y=2x+3$ at the point (−7,−11)

d $y=3x^2-5x-2$ and $y=x^2-2x+3$ at the point (−1, 6)

e $y=\sqrt{x}$ and $y=2x-15$ at the point (9, 3)

7 A curve is given by $y=x^3+5x^2-8x+1$
Which of the following statements are true?

a $\frac{dy}{dx}>4$ when $x=1$

b $\frac{dy}{dx}<0$ when $x=0$

c $\frac{dy}{dx}=0$ when $x=4$

d $\frac{dy}{dx}>0$ when $x=-2$

e $\frac{dy}{dx}$ when $x=-4$ is equal to $\frac{dy}{dx}$ when $x=\frac{2}{3}$

f $\frac{dy}{dx}$ when $x=-1$ is equal to $\frac{dy}{dx}$ when $x=1$

Reasoning and problem-solving

Strategy

To solve differentiation problems involving polynomials with rational powers

1. Use the laws of algebra to make your expression the sum of terms of the form ax^n
2. Apply $f(x)=ax^n \Rightarrow f'(x)=nax^{n-1}$ to each term to find the derivative.
3. Substitute any numbers required and answer the question.

Example 3

Given that $f(x)=\frac{x+1}{\sqrt{x}}$, find the expression $f'(x)$ and hence find $f'(4)$

$$f(x)=\frac{x+1}{\sqrt{x}}$$

Write in index form

$$=\frac{x}{x^{\frac{1}{2}}}+\frac{1}{x^{\frac{1}{2}}}$$

$$f(x)=x^{\frac{1}{2}}+x^{-\frac{1}{2}}$$

Divide by $x^{\frac{1}{2}}$ to make your expression a sum of terms of the form ax^n (1)

$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}x^{-\frac{3}{2}}$$

$$f'(x)=\frac{1}{2x^{\frac{1}{2}}}-\frac{1}{2x^{\frac{3}{2}}}$$

$$f'(x)=\frac{1}{2\sqrt{x}}-\frac{1}{2(\sqrt{x})^3}$$

Apply $f'(x)=nax^{n-1}$ to each term. (2)

$$f'(4)=\frac{1}{2\sqrt{4}}-\frac{1}{2(\sqrt{4})^3}$$

Substitute $x=4$ (3)

$$f'(4)=\frac{1}{4}-\frac{1}{16}$$

$$=\frac{3}{16}$$

Exercise 4.2B Reasoning and problem-solving

1 Work out the derivative with respect to x of

a $f(x)=x(x+1)$

b $g(x)=(x-1)(x+1)$

c $h(x)=x^2(1-3x)$

d $k(x)=\sqrt{x}(x+3)$

e $m(x)=3x(2x^2+x-3)$

f $n(x)=x(\sqrt{x}+1)$

g $p(x)=x^{-1}(4+2x)$

h $q(x)=(x^{-1}-2)(x+1)$

i $r(x)=\frac{1}{x}(x^2+3x+1)$

2 Given y, find $\frac{dy}{dx}$

a $y=x$

b $y=x+\sqrt{x}$

c $y=\frac{x+\sqrt{x}}{x}$

d $y=\frac{1-x-2\sqrt{x}}{x}$

e $y=\frac{x-x^2}{\sqrt{x}}$

f $y=\frac{1+2\sqrt{x}}{x}$

g $y=\frac{x^2+3x-1}{x}$

h $y=\frac{1}{x}+\frac{1}{x^2}$

i $y=\frac{1}{\sqrt[3]{x}}$

3 Work out the derivative with respect to x of each of these functions.

a $f(x)=(x-3)(x+1)$

b $g(x)=(x-4)(x-5)$

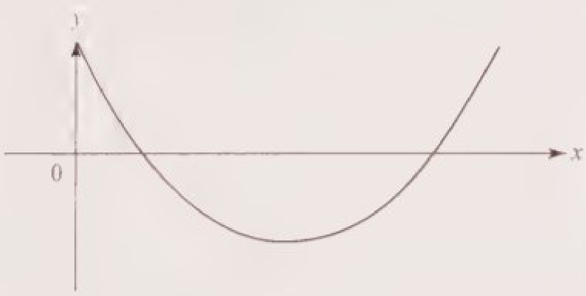
c $h(x)=(1-x)(3+x)$

d $j(x)=(x+1)(2x+1)$

e $k(x)=x(x+1)(x-1)$

f $m(x)=(x+1)(\sqrt{x}+1)$

4 The sketch shows part of the curve $y=(x-1)(x-7)$ near the origin.



a Work out $\frac{dy}{dx}$

b Identify where the curve crosses the x-axis.

c Work out the point where the gradient of the curve is zero.

d i Where is $\frac{dy}{dx}<0$?

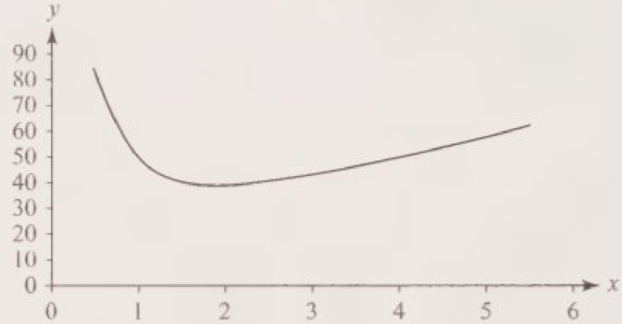
ii Where is $\frac{dy}{dx}>0$?

e Choose the correct word to make each statement true.

i When $\frac{dy}{dx}<0$, the curve is (rising/falling) from left to right.

ii When $\frac{dy}{dx}>0$, the curve is (rising/falling) from left to right.

5 The sketch shows the curve $y=\frac{10x^2+40}{x}$, $0.5\le x\le 5.5$



a Work out $\frac{dy}{dx}$

b i For which value of x is the gradient of the curve zero? Show your working.

ii What is the value of y at this point?

c For which values of x is the curve

i Decreasing, ii Increasing?

Challenge

6 Prove that the gradient of the function $f(x)=(x+1)^4$ at the point (1, 16) is 32

See Ch2.2

For a reminder on expanding brackets and the binomial theorem.

7 a Differentiate the function $f(x)=(x-1)(x+2)$

b Work out the gradient of the curve $y=f(x)$ at $x=5$

c At which point is the gradient equal to zero?

d A line with equation $y=2x-k$, where k is a constant, is a tangent to the curve.

i At what point does the tangent touch the curve?

ii What is the value of k?

4.3 Rates of change

Fluency and skills

When y is a function of x, the gradient of a graph of y against x tells you how the y-measurement is changing per unit x-measurement.

Key point

The rate of change of y with respect to x can be written $\frac{dy}{dx}$

See Ch7.2 For more on distance–time graphs.

The gradient of a distance-time graph is a measure of the rate of change of distance (r) with respect to time (t), this is called **velocity** (v).

Key point

$$v = \frac{dr}{dt}$$

If v metres per second represents velocity and t seconds represents time, then the gradient, $\frac{dv}{dt}$, is a measure of the rate at which velocity is changing with time, in metres per second per second.

Key point

The rate of change of velocity is called **acceleration**, $a = \frac{dv}{dt}$

See Ch7.4 For more on acceleration as a derivative.

Acceleration is the derivative of a derivative, which is called the **second derivative**.
A similar notation is used for the second derivative, $f''(x)$, as for the first derivative, $f'(x)$

Key point

$$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) \Rightarrow \frac{d^2y}{dx^2} = f''(x)$$

Example 1

A particle is moving on the y-axis such that its distance, r cm, from the origin is given by $r = t^3 + 2t^2 + t$, where t is the time measured in seconds.

a Use the fact that the velocity $v = \frac{dr}{dt}$ to find an expression for the particle's velocity.

b Use the fact that the acceleration $a = \frac{dv}{dt} = \frac{d^2r}{dt^2}$ to find an expression for the particle's acceleration.

a $r = t^3 + 2t^2 + t$ $\qquad \frac{dr}{dt} = 3t^2 + 4t + 1$ — Use $f'(t) = nat^{n-1}$

b $\frac{dr}{dt} = 3t^2 + 4t + 1$ $\qquad \frac{d^2r}{dt^2} = 6t + 4$

PURE

By differentiating a function and sketching a graph of the differential, you can find out whether

the function is increasing $\left(\frac{dy}{dx} > 0\right)$,

the function is decreasing $\left(\frac{dy}{dx} < 0\right)$,

or the function is neither increasing nor decreasing $\left(\frac{dy}{dx} = 0\right)$, in which case you call it **stationary**.

A positive gradient means an increasing function. A negative gradient means a decreasing function.

Example 2

Consider the curve $y = 2x^3 - 3x^2 - 36x + 2$

a Work out $\frac{dy}{dx}$

b Use your equation from part **a** to sketch a graph of $\frac{dy}{dx}$. You must show your working.

c Using your sketch, determine where the value of y is increasing and decreasing.

d Work out where the value of y is stationary.

See Ch2.4
For a reminder on curve sketching.

a $y = 2x^3 - 3x^2 - 36x + 2$

$\frac{dy}{dx} = 6x^2 - 6x - 36$

Differentiate to find the gradient function.

b $\frac{dy}{dx} = 6(x^2 - x - 6)$

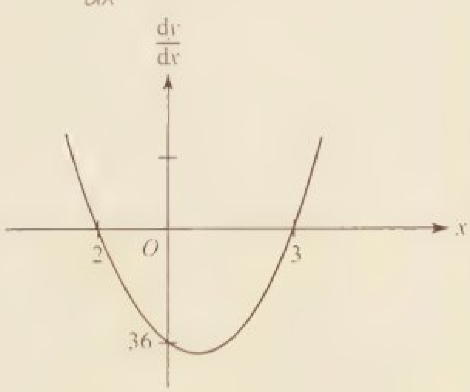
$= 6(x-3)(x+2)$

Factorise.

When $x = 0$, $\frac{dy}{dx} = 6(-3)(2) = -36$

When $\frac{dy}{dx} = 0$, $x = 3$ or -2

Determine where the gradient function crosses the axes.



Use this information to sketch a graph. You can also use your calculator to sketch the function.

c From the sketch, $\frac{dy}{dx} > 0$ (it's increasing) when $x < -2$ or when $x > 3$

From the sketch, $\frac{dy}{dx} < 0$ (it's decreasing) when $-2 < x < 3$

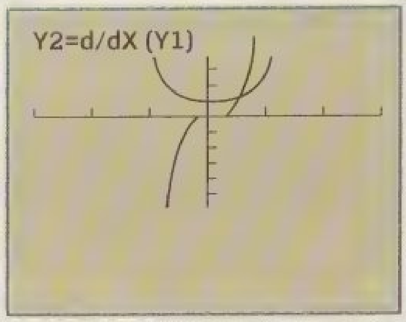
d The value of y will be neither increasing nor decreasing (it is stationary) at both $x = -2$ and $x = 3$

Interpret the graph.

Calculator

Try it on your calculator

You can use a calculator to sketch a function and its derivative.



Activity

Find out how to sketch the curve $f(x) = 3x^4 - x^3 + 1$ and its derivative $f'(x)$ on *your* calculator.

Exercise 4.3A Fluency and skills

1 Calculate the rate of change of the following functions at the given points. You must show all your working.

- **a** $f(x) = x^2 + 10x$ at $x = 4$
- **b** $g(x) = x^3 + 2x^2 + x + 1$ at $x = -1$
- **c** $h(x) = 5x + 6$ at $x = 1$
- **d** $k(x) = x + \frac{1}{x}$ at $x = 3$
- **e** $m(x) = 9x^2 + 6x + 1$ at $x = 1$
- **f** $n(x) = x^{-1} + x^{-2}$ at $x = -2$
- **g** $p(x) = \frac{1}{\sqrt{x}}$ at $x = \frac{1}{9}$
- **h** $q(x) = x^{\frac{3}{2}} + 1$ at $x = 36$
- **i** $r(x) = x^4 - 8x^2$ at $x = -2$
- **j** $s(x) = \sqrt{x} - x$ at $x = 4$
- **k** $t(x) = 2\pi$ at $x = 7$
- **l** $u(x) = 4 - 3x$ at $x = -10$
- **m** $v(x) = \frac{1}{2x^4}$ at $x = -1$
- **n** $w(x) = 1 - 3x - 2x^2$ at $x = 0$
- **o** $y(x) = \frac{162}{x^2} + 2x^2$ at $x = 3$
- **p** $z(x) = 20\sqrt{x} + \frac{1000}{\sqrt{x}}$ at $x = 25$

2 Work out the rate of change of the rate of change, $\left(\frac{d^2y}{dx^2}\right)$, of the following functions at the given points. You must show all your working.

- **a** $y = x^3 + x$ at $x = 3$
- **b** $y = \frac{10}{x}$ at $x = 2$
- **c** $y = \frac{1}{\sqrt{x}}$ at $x = 1$
- **d** $y = x^{1} - x^{2}$ at $x = -2$
- **e** $y = x^2 - \frac{4}{x}$ at $x = 2$
- **f** $y = x^3 + 4x^2 + 3x$ at $x = -1$
- **g** $y = \sqrt{x}$ at $x = 0.25$
- **h** $y = 3x - 4\sqrt{x}$ at $x = 0.16$
- **i** $y = 4 - ax$, where a is a constant, at $x = -3$
- **j** $y = x + 2 + \frac{1}{x}$ at $x = \frac{1}{2}$
- **k** $y = \frac{1}{x^4}$ at $x = 0.5$
- **l** $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}$ at $x = -2$

3 Using the gradient function of each curve, determine where the curve is

- **i** Stationary,
- **ii** Increasing,
- **iii** Decreasing.

a $y=x^2+4x-12$

b $y=3-5x-2x^2$

c $y=x^2-1$

d $y=\frac{x^3}{3}-4x+1$

e $y=1+21x-2x^2-\frac{1}{3}x^3$

f $y=\frac{x^3}{3}-\frac{3x^2}{2}+2x-1$

g $y=\frac{2}{3}x^{\frac{3}{2}}, x>0$

h $y=1-\frac{1}{x}, x>0$

i $y=\frac{x^3}{3}$

j $y=\frac{x^4}{4}$

4 By finding expressions for $\frac{dy}{dx}$, determine which of each pair of functions has the greater rate of change with respect to x at the given x-value.

a $y=x^2$ and $y=-x^2$ at $x=-1$

b $y=5-3x$ and $y=\frac{1}{x}$ at $x=0.5$

c $y=x^2-2$ and $y=x^2-x-2$ at $x=3$

d $y=1+10x-x^3$ and $y=x-x^2$ at $x=2$

Reasoning and problem-solving

Strategy

To answer a problem involving rate of change

1. Read and understand the context, identifying any function or relationship.
2. Differentiate to get a formula for the relevant rate of change.
3. Evaluate under the given conditions.
4. Apply this to answer the initial question, being mindful of the context and units.

Example 3

A conical vat is filled with water. The volume in the vat at time t seconds is Vml.

V is a function of t such that the volume at time t is found by multiplying the cube of t by 50π

a Work out $V'(t)$, the rate at which the vat is filled, in millilitres per second.

b Calculate the rate at which the vat is filling with water after 3 seconds.

a $V(t)=50\pi t^3$ — (1) Identify the function.

$V'(t)=150\pi t^2$ — (2) Differentiate to get a formula for the relevant rate of change.

b $V'(3)=150\pi\times 3^2$

$=1350\pi$

$V'(3)\approx 4241$ — (3) Evaluate under the required conditions.

After 3 seconds, the vat is filling at the rate of 4241 ml s^{-1} (4 sf) — (4) Answer the question in context.

Example 1

An object is thrown into the air. Its height after t seconds is given by $h = 1 + 30t - 5t^2$ where h is its height in metres.

a Write down an expression for the rate at which the object is climbing, in metres per second.

b Work out

i When the object is rising,

ii When the object is falling.

c After how many seconds does the object reach its maximum height?

a $h = 1 + 30t - 5t^2$ — (1) Identify the function.

$\frac{dh}{dt} = 30 - 10t$ — (2) Differentiate to get a formula for the rate of change.

b i $\frac{dh}{dt} > 0 \Rightarrow 30 - 10t > 0$ — (3) Evaluate when the rate of change is greater than zero.

$30 > 10t$

$t < 3$ s

The object is rising before the third second. — (4) Answer the question, remembering units.

ii $\frac{dh}{dt} < 0 \Rightarrow 30 - 10t < 0$

$30 < 10t$

$t > 3$ s

The object is falling after the third second.

c $\frac{dh}{dt} = 0 \Rightarrow 30 - 10t = 0$

$30 = 10t$

$t = 3$ s

At exactly three seconds the object has reached the top of its flight and is neither rising nor falling.

Exercise 4.3B Reasoning and problem-solving

1 A taxi driver charges passengers according to the formula $C(x) = 2.5x + 6$, where C is the cost in £ and x is the distance travelled in km.

Work out the rate in £ per km that the taxi driver charges.

2 A block of ice melts so that its dry mass M, at a time t minutes after it has been removed from the freezer, is given by $M(t) = 500 - 7.5t$

a Work out $\frac{dM}{dt}$, the rate at which the mass is changing.

b Your answer should have a negative sign in it. Explain why this is.

3 The volume of a sphere is $V = \frac{4}{3}\pi r^3$

The surface area of a sphere is $S = 4\pi r^2$

A spherical bubble is expanding.

a Find an expression for the rate of change of the volume as r increases.

b Calculate the rate of change of the volume of the bubble when its radius is 3 cm.

c Calculate the rate of change of the surface area with respect to the radius when the radius is 3 cm.

4 A parabola has equation $y = 35 - 2x - x^2$

a Work out the rate of change of y with respect to x when x is equal to

i -2 ii 2 iii 5

b When will this rate of change be zero?

5 A skydiver jumps from an ascending plane. His height, h m above the ground, is given by $h = 4000 + 3t - 4.9t^2$, where t seconds is the time since leaving the plane.

a Work out $\frac{dh}{dt}$, the rate at which the skydiver is falling.

b How fast is he falling after 5 seconds?

c How fast is he falling after 10 seconds?

d Calculate his acceleration at this time.

6 A golf ball was struck on the moon in 1974. Its height, h m, is modelled by $h = 10t - 1.62t^2$, where t seconds is the time since striking the ball.

a Calculate $\frac{dh}{dt}$, the rate at which the height of the ball is changing.

b After how much time will the ball be falling?

c Calculate the acceleration of the ball in the gravitational field of the moon.

7 A cistern fills from empty. A valve opens and the volume of water, V ml, in the cistern t seconds after the valve opens is given by $V = 360t - 6t^2$

a Write down an expression for the rate at which the cistern is filling after t seconds.

b Calculate the rate at which the cistern is filling after

i 10 seconds, ii 20 seconds.

c When the rate is zero, a ballcock shuts off the valve. At what time does this occur?

d What volume of water is in the tank when the valve closes?

8 For each function

i Work out an expression for the rate of change of y with respect to x

ii Evaluate this rate of change when $x = 0$

iii By expressing the rate of change in the form $\frac{dy}{dx} = (x+a)^2 + b$, establish that each function is increasing for all values of x

a $y = \frac{x^3}{3} + 5x^2 + 30x + 1$

b $y = \frac{x^3}{3} - 3x^2 + 12x - 4$

c $y = \frac{x^3}{3} - \frac{5}{2}x^2 + 8x + \frac{1}{2}$

9 The derivative of a function is

$\frac{dy}{dx} = 8x - x^2 - 17$

Show that the function is always decreasing.

Challenge

10 A curve has equation

$y = x^4 + \frac{1}{x^2}, x \geq 1$

a Work out i $\frac{dy}{dx}$ ii $\frac{d^2y}{dx^2}$

b Show that the gradient of the curve is an increasing function.

4.4 Tangents and normals

Fluency and skills

See Ch1.6 For a reminder on the equation of a straight line.

When lines with gradient m_1 and m_2 are **perpendicular** to each other, $m_1 \times m_2 = -1$

Key point

$m_1 = -\frac{1}{m_2}$ for perpendicular lines.

The **tangent** to the curve $y = f(x)$, which touches the curve at the point $(x, f(x))$, has the same gradient as the curve at that point, giving $m_T = f'(x)$

The **normal** to the curve $y = f(x)$, which passes through the point $(x, f(x))$, is perpendicular to the tangent at that point.

giving $m_N = -\frac{1}{m_T} = -\frac{1}{f'(x)}$

> A line with gradient m passing through the point (a, b) has equation $(y - b) = m(x - a)$

Example 1

A curve has equation $y = 2x^2 - 3x - 10$

a Work out the equation of the tangent to the curve at the point (4, 10)

b Work out the equation of the normal to the curve when $x = -2$

a $y = 2x^2 - 3x - 10$ so $\frac{dy}{dx} = 4x - 3$ — Differentiate

At the point (4, 10) the tangent has gradient

$\frac{dy}{dx} = 4 \times 4 - 3 = 13$ — Substitute.

The equation of the tangent is

$(y - 10) = 13(x - 4)$ — Use $(y - b) = m(x - a)$

$y = 13x - 42$

b When $x = -2$

$y = 2 \times (-2)^2 - 3 \times (-2) - 10$ — Substitute $x = -2$ into the original equation.

$y = 4$

So (−2, 4) is a point on the normal.

At (−2, 4) the tangent has gradient

$\frac{dy}{dx} = 4 \times (-2) - 3 = -11$

So the normal has a gradient of $\frac{1}{11}$ — Use $m = -\frac{1}{m}$

The equation of the normal is

$(y - 4) = \frac{1}{11}(x + 2)$ — Use $(y - b) = m(x - a)$

$11y - x = 46$

Exercise 4.4A Fluency and skills

1 Work out the equation of the tangent to each of these curves at the given points. Show your working.

a $y = 2x^2 + 3x - 1$ at $(1, 4)$

b $y = 3 - x - x^2$ at $(-2, 1)$

c $y = 2x^3 + 3x^2$ at $(1, 5)$

d $y = 5x^{-2} - x^{-1}$ at $(2, 4)$

e $y = \frac{3}{x}$ at $(3, 1)$

f $y = \frac{16}{x^2}$ at $(4, 1)$

g $y = \sqrt{x}$ at $(9, 3)$

h $y = \frac{25}{\sqrt{x}}$ at $(25, 5)$

i $y = \sqrt{x} + \frac{2}{\sqrt{x}}$ at $(4, 3)$

j $y = \frac{1}{\sqrt{x}} + \frac{1}{x}$ at $(1, 2)$

2 Work out the equation of the normal to each curve at the given points. Show your working.

a $y = x^2 + 2x - 7$ at $(2, 1)$

b $y = 4 - 5x - x^2$ at $(-3, 10)$

c $y = 3 - x^3$ at $(2, -5)$

d $y = x^4 + 2x^3 + x^2$ at $(1, 4)$

e $y = \frac{6}{x}$ at $(2, 3)$

f $y = \sqrt{x} + x$ at $(4, 6)$

g $y = \frac{1}{x} + \frac{1}{\sqrt{x}}$ at $(4, \frac{3}{4})$

h $y = \frac{1}{x^2} + \frac{\sqrt{x}}{x^2}$ at $(1, 2)$

3 A curve has equation $y = 3x^2$

a Work out the point on the curve with x-coordinate 3

b Work out the gradient of the curve at this point. Show your working.

c Work out the equation of the tangent to the curve at this point.

4 A line is a tangent to the curve $y = x^2 + 3x - 1$ at the point $(1, k)$

a What is the value of k?

b What is the gradient of the tangent at this point? Show your working.

c Work out the equation of the line.

5 A curve has equation $y = x^2 + 7x - 9$

a Calculate the point on this curve with x coordinate 1

b Calculate the gradient of the tangent to the curve at this point. Show your working.

c Hence state the gradient of the normal to the curve at this point.

d Work out the equation of the normal to the curve at this point.

6 A curve has equation $y = 2x + \frac{2}{x}$

a Calculate the point on the curve with x-coordinate 2

b Calculate the gradient of the tangent to the curve at this point. Show your working.

c Hence state the gradient of the normal to the curve at this point.

d Work out the equation of the normal to the curve at this point.

7 A parabola has equation $y = x^2 + 6x + 5$

a i Work out the gradient of the tangent at the point where $x = -3$. Show your working.

ii Give the equation of the tangent.

b Give the equation of the normal to the curve at this point.

c In a similar way, work out the equation of the tangent and normal to each curve at the given point. Show your working.

i $y = x^2 + 2x - 24$ at $x = -1$

ii $y = x^2 + 10x$ at $x = -5$

iii $y = 21 + 4x - x^2$ at $x = 2$

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Reasoning and problem-solving

Strategy 1

To work out where a tangent or normal meets a curve

1. Differentiate the function for the curve.
2. Equate this to the gradient of the tangent or normal (remember $m_T = -\frac{1}{m_N}$).
3. Rearrange and solve for x
4. Substitute x in the function and solve for y

Example 2

The line $y = 3x + b$ is a tangent to the curve $y = 2x + 4\sqrt{x}$, $x > 0$

a Work out the point where the tangent meets the curve, thus find the value of the constant b

b Work out the equation of the normal to the curve at this point.

a $y = 2x + 4\sqrt{x} = 2x + 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2 + \frac{1}{2} \times 4x^{-\frac{1}{2}}$ — (1) Differentiate the function of the curve.

$= 2 + \frac{2}{\sqrt{x}}$

$m_{tan} = 3$

so $2 + \frac{2}{\sqrt{x}} = 3$ — (2) Set $\frac{dy}{dx}$ equal to the gradient of the tangent.

$\sqrt{x} = 2$ — (3) Rearrange and solve for x

$x = 4$

x cannot be negative.

When $x = 4$,

$y = 2 \times 4 + 4 \times \sqrt{4} = 16$ — (4) Solve for y

So the tangent touches the curve at (4, 16)

$3(4) + b = 16$

$b = 4$

b $m_N = -\frac{1}{m_T} = -\frac{1}{3}$ — Find the gradient of the normal.

$y - 16 = -\frac{1}{3}(x - 4)$ — Use $y - b = m(x - a)$

$y = \frac{52}{3} - \frac{x}{3}$

Strategy 2

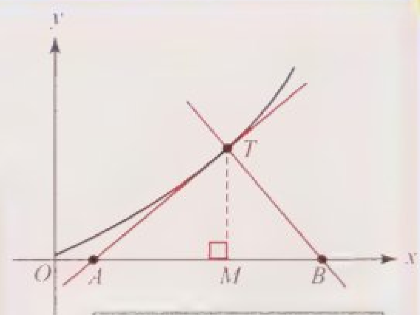
To work out the area bound between a tangent, a normal and the x-axis/y-axis

1. Work out the equation of the tangent and, from it, the equation of the normal.
2. Work out where each line crosses the required axis. Lines cut the x-axis when $y = 0$, and the y-axis when $x = 0$
3. Sketch the situation if required.
4. Use $A = \frac{1}{2} \times$ base $\times$ height, where the base is the length between the intercepts on the x-axis or y-axis and height is the y-coordinate or x-coordinate respectively.

Example 3

The point $T(1, 2)$ lies on the curve $y = x^3 + x$

Work out the triangular area trapped between the tangent and the normal to the curve at this point and the x-axis.



The gradient of the tangent at $T(1, 2)$ is

$\frac{dy}{dx} = 3x^2 + 1 = 3 + 1$

$= 4$

So the gradient of the normal is $-\frac{1}{4}$

$(y - b) = m(x - a)$

$y - 2 = 4(x - 1)$

This crosses the x-axis when $y = 0$

So, $0 - 2 = 4(x - 1)$

$x = \frac{1}{2}$

Therefore, A is the point $\left(\frac{1}{2}, 0\right)$

Equation of the normal is $y - 2 = -\frac{1}{4}(x - 1)$

This crosses the x-axis when $y = 0$

So, $0 - 2 = -\frac{1}{4}(x - 1)$

$x = 9$

B is the point $(9, 0)$

So $AB = 9 - \frac{1}{2} = 8\frac{1}{2}$

$MT = x_T = 2$

Area of triangle $= \frac{1}{2} \times AB \times MT$

$= \frac{1}{2} \times 8\frac{1}{2} \times 2 = 8\frac{1}{2}$ (square units)

Substitute the x-coordinate of the point. Remember you could do this using your calculator.

Use $m_n = -\frac{1}{m_t}$

Use $(y - b) = m(x - a)$ to work out the equation of the tangent.

(1) Work out the equation of the tangent.

(2) Work out where each line crosses the required axis.

(1) Use the equation of the tangent to work out the equation of the normal.

(4) Use Area $= \frac{1}{2} \times$ base $\times$ height

Exercise 4.4B Reasoning and problem-solving

1 The line with equation $y=1-3x$ is a tangent to the curve $y=x^2-7x+k$ where k is a constant.

   a Calculate the value of x at the point where the tangent meets the curve.

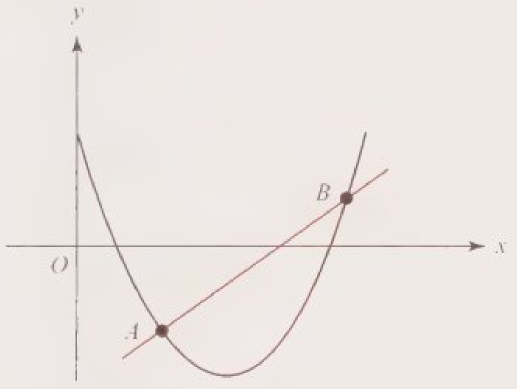
   b Hence calculate the value of k

2 The curve $y=x+\frac{3}{x}, x>0$ has a normal that runs parallel to the line

   $y=3x+4$

   Work out the point where the normal crosses the curve at right angles.

3 a Work out the equation of the normal to the curve $y=x^2-3x+1$ at the point A, where $x=1$



   b Work out the point B where the normal at A crosses the curve again.

   c Prove that the line AB is *not* normal to the curve at B

4 The curves $y=\frac{2x^2+1}{2}$ and $y=\frac{1+4x}{x}, x\neq0$ intersect at the point (a, b). At this point, the line that is the tangent to one curve is the normal to the other line.

   a Use two methods to work out an expression for the gradient of this line.

   b Work out the point (a, b)

   c Work out the equation of the line.

5 a Work out the equation of the tangent to the curve $y=2-\frac{9}{x}$ when $x=3$

   b Work out the equation of the normal to the curve at the same point.

   c Calculate the area of the triangle bound by the tangent, the normal and the y-axis.

6 A cubic curve $y=x^3+x^2+2x+1$ has a tangent at $x=0$

   a Work out the equation of this tangent.

   b Work out the coordinates of the point B where the tangent crosses the curve again.

   c Work out the coordinates of the point C where the tangent at B crosses the x-axis.

   d Calculate the area of the triangle BCO where O is the origin.

7 A parabola has the equation $y=2x^2-3x+1$

   a Work out the equation of the tangent to the curve that is parallel to the line $y=5x$

   b Work out the equation of the normal at this point.

8 A point in the first quadrant (p, q) lies on the curve $y=x^3+1$

   The tangent at this point is perpendicular to the line $y=-\frac{x}{12}$

   a Calculate the values of p and q

   b What is the equation of the tangent at this point?

   c What is the equation of the normal at this point?

9 A normal to the curve $y=x+\frac{18}{x}, x>0$, is parallel to $y=x$

   a Work out the coordinates of the point where the normal crosses the curve at right angles.

   b Work out the equation of the tangent at this point.

10 For each parabola

i Express the equation in the form $b \pm (x+a)^2$

ii Hence deduce the coordinates of the turning point on the parabola.

iii Work out the equation of the tangent and the normal at this point.

iv Comment on your answers.

a $y = x^2 + 6x - 1$

b $y = x^2 - 10x + 5$

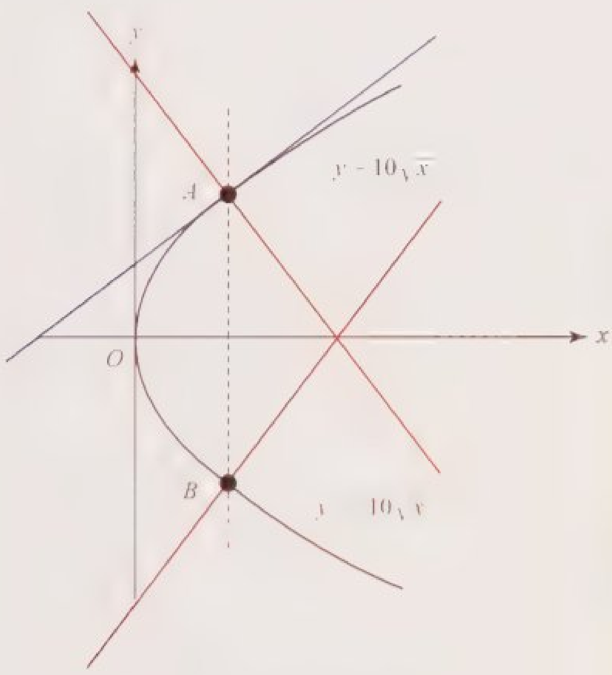
c $y = x^2 - 3x - 7$

d $y = 4 - 8x - x^2$

e $y = x^2 + 4$

f $y = 1 - 2x - x^2$

11 A parabolic mirror has a cross-section as shown.



The branch above the x-axis has equation $y = 10\sqrt{x}$

The branch below it has equation $y = -10\sqrt{x}$

The line AB is parallel to the y-axis.
Let $x_A = x_B = p$

a Work out the equation of the normal to the curve

i At A ii At B

b Show that these two normals intersect on the x-axis.

12 The parabola $y = 4 + 2x - 2x^2$ crosses the y-axis at the point (p, q)

a State the values of p and q

b Work out the gradient of the tangent of the curve at this point.

c Work out the equation of the normal to this point.

d The curve crosses the x-axis at two points.

i Work out the coordinates of these points.

ii Work out the equation of the normal at both points.

e A related curve, $y = 4 - 2x^2$ crosses the y-axis at $(0, 4)$

Work out the equation of the normal to this curve at this point.

Challenge

13 a Work out the equation of the tangent to the curve $y = \frac{360}{x}$ $(x \geq 1)$ at the point where $x = 30$

b Work out the equation of the normal at the point where $x = 60$

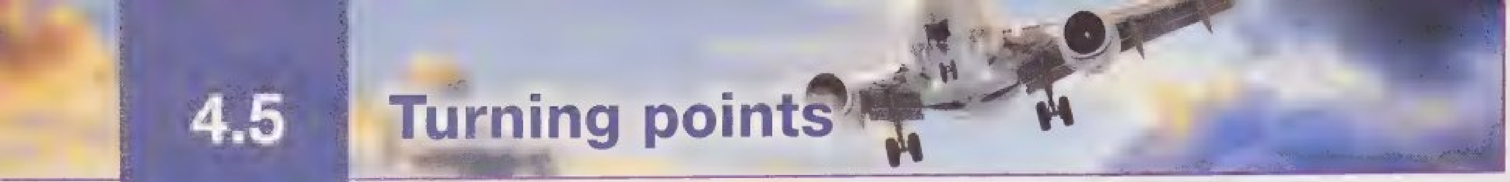
c i At what point is the gradient of the normal $\frac{1}{10}$?

ii Give the equation of this normal.

d The line $y = -40x + k$ is a tangent to the curve.

i At what point does this line touch the curve?

ii What is the value of k?



4.5 Turning points

Fluency and skills

When a curve changes from an increasing function to a decreasing function or vice versa, it passes through a point where it is stationary. This is called a **turning point** or **stationary point**.

> **Key point**
> At a turning point, the gradient of the tangent is zero. Therefore, you can work out the coordinates of the turning point by equating the derivative to zero.

A turning point is a stationary point, but a stationary point is not necessarily a turning point. You will learn about other types of stationary point in **Section 15.1**

Here are examples of a **maximum** turning point and a **minimum** turning point.

At a maximum turning point, as x increases, the gradient changes from positive through zero to negative.

At a minimum turning point, as x increases, the gradient changes from negative through zero to positive.

Example 1

Work out the coordinates of the turning point on the curve $y = x^2 + 4x - 12$ and determine its nature by inspection of the derivative either side of the point. Show your working.

$y = x^2 + 4x - 12$ so $\frac{dy}{dx} = 2x + 4$ — Differentiate.

At a turning point,

$\frac{dy}{dx} = 0 \Rightarrow 2x + 4 = 0 \Rightarrow x = -2$ — Find the value of x when the derivative is equal to zero.

$y = (-2)^2 + 4 \times (-2) - 12 \Rightarrow y = -16$ — Substitute into the original equation.

The turning point is at $(-2, -16)$

At $x = -2.1$, $\frac{dy}{dx} = 2(-2.1) + 4 = -0.2$

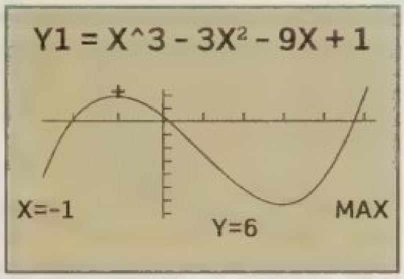
At $x = -1.9$, $\frac{dy}{dx} = 2(-1.9) + 4 = 0.2$ — Consider the value of the derivative either side of the turning point.

The gradient is increasing from negative to positive, so the point $(-2, -16)$ is a minimum turning point.

Calculator

Try it on your calculator

You can find turning points on a graphics calculator.



Activity

Find out how to find the turning points of $y = x^3 - 3x^2 - 9x + 1$ on *your* graphics calculator.

As well as by inspection, the nature of a turning point can be determined by finding the second derivative with respect to x, $\frac{d^2y}{dx^2}$

If the gradient, $\frac{dy}{dx}$, is *decreasing* and the second derivative is negative, the turning point is a *maximum*.

Key point

At a *maximum* turning point, $\frac{d^2y}{dx^2} < 0$

If the gradient, $\frac{dy}{dx}$, is *increasing* and the second derivative is positive, the turning point is a *minimum*.

Key point

At a *minimum* turning point, $\frac{d^2y}{dx^2} > 0$

Example 2

Use calculus to work out the coordinates of the turning point on the curve $y = x + \frac{1}{x}, x > 0$ and determine its nature.

$y = x + \frac{1}{x}$ so $\frac{dy}{dx} = 1 - \frac{1}{x^2}$

At a turning point, $\frac{dy}{dx} = 0$ so $1 - \frac{1}{x^2} = 0$

Solving for x gives $x = \pm 1$

but $x > 0$, so $x = 1$

When $x = 1$,

$y = 1 + \frac{1}{1} = 2$ — Substitute into the original equation.

So the turning point is at $(1, 2)$

$\frac{dy}{dx} = 1 - \frac{1}{x^2} = 1 - x^{-2}$

$\frac{d^2y}{dx^2} = \frac{2}{x^3}$ — Find the second derivative to determine the nature of the turning point.

When $x = 1$,

$\frac{d^2y}{dx^2} = \frac{2}{1^3} = 2$

The second derivative is positive, so the turning point is a minimum.

Exercise 4.5A Fluency and skills

1 For each curve, work out the coordinates of the stationary point(s) and determine their nature by inspection. Show your working.

a $y=x^2+4x-5$ b $y=x^2+4x-32$

c $y=x^2-6x-7$ d $y=1-x^2$

e $y=2x^2+7x+6$ f $y=20-x-x^2$

g $y=6x^2-x-1$ h $y=2-13x-7x^2$

i $y=6-x-2x^2$ j $y=x+\frac{4}{x}, x>0$

k $y=2x+\frac{18}{x}, x\neq 0$

l $y=10-x-\frac{1}{x}, x\neq 0$

m $y=x-10\sqrt{x}, x\geq 0$

n $y=x^2-32\sqrt{x}, x\geq 0$

2 The curve $y=x^3-6x^2$ has two turning points.

a Work out the coordinates of both turning points. Show your working.

b Use the second derivative to determine the nature of each.

3 Work out the coordinates of the turning points of $y=2x^3+30x^2+1$ and determine their nature. Show your working.

4 $f(x)=2x^3-9x^2+12x+7$

a Differentiate $f(x)$

b The curve $y=f(x)$ has two turning points. Work out the coordinates of them both and determine their natures. Show your working.

c Repeat this process with the following functions.

i $f(x)=x^3-3x^2-24x+1$

ii $f(x)=x^3+3x^2-45x-45$

iii $f(x)=1-36x-21x^2-2x^3$

iv $f(x)=2x^3-11x^2-8x+2$

v $f(x)=3-4x+5x^2-2x^3$

vi $f(x)=5+x-2x^2-4x^3$

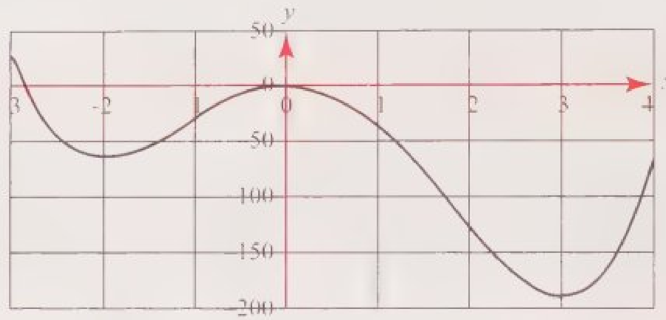
5 The function $f(x)=3x^4+8x^3-6x^2-24x-1$ has three turning points.

a Show that stationary points can be found at $x=1$, $x=-1$ and $x=-2$

b Work out the coordinates of the three stationary points.

c Use the second derivative to help you establish the nature of each.

6 The function $f(x)=3x^4-4x^3-36x^2$ has three turning points. A sketch of the graph of the function is shown.



a The sketch suggests that these turning points can be found at $x=-2$, $x=0$ and $x=3$
Show that this is the case.

b Use the second derivative to verify the nature of each turning point.

7 A function is defined by $f(x)=8x+\frac{72}{x}, x>0$

a Show that the *only* stationary point on the curve $y=f(x)$ is at $x=3$

b State the coordinates of the stationary point and determine its nature. Show your working.

c A related function is defined by $f(x)=8x+\frac{72}{x}, x\neq 0$
It has a stationary point at $x=-3$
Determine the nature of this stationary point. Show your working.

Reasoning and problem-solving

Strategy 1

To identify the main features of a curve

1. Work out where it crosses the axes ($x = 0$ and $y = 0$)
2. Consider the behaviour of the curve as x tends to infinity and identify any asymptotes.
3. Work out the coordinates of the turning points $\left(\frac{dy}{dx} = 0\right)$ and determine their nature.
4. Use the information you have found to sketch the function.

See Ch2.4 For a reminder about sketching curves

Example 3

$y = (x - 10)(x + 5)(x + 14)$

a Show that the curve crosses the y-axis at $(0, -700)$

b Show that there is a maximum turning point at $(-10, 400)$ and a minimum turning point at $(4, -972)$

c Sketch the curve.

a The curve crosses the y-axis when $x = 0$

So $y = -10 \times 5 \times 14$

$= -700$, giving $(0, -700)$ as a point on the curve.

(1) Find where the graph crosses the axes.

b $y = (x-10)(x+5)(x+14)$

$y = x^3 - 9x^2 - 120x - 700$

Expand.

$\frac{dy}{dx} = 3x^2 - 18x - 120 = 3(x-4)(x+10)$

Differentiate and factorise.

Stationary points occur when $\frac{dy}{dx} = 0 \Rightarrow x = 4$ and $x = -10$

When $x = 4$

$y = (4 - 10)(4 + 5)(4 + 14) = -972$, giving $(4, -972)$

When $x = -10$

$y = (-10 - 10)(-10 + 5)(-10 + 14) = 400$, giving $(-10, 400)$

(3) Work out the coordinates of the turning points.

$\frac{dy}{dx} = 3x^2 - 18x - 120$

So $\frac{d^2y}{dx^2} = 6x - 18$

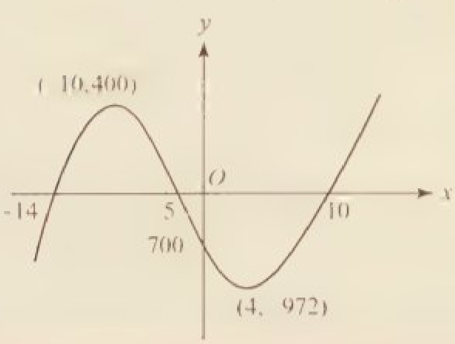
(3) Find the second derivative.

When $x = 4$, the second differential is positive so $(4, -972)$ is a minimum turning point.

When $x = -10$, it is negative so $(-10, 400)$ is a maximum turning point.

(3) Use the second derivative to determine the nature of the turning points.

c



(4) Use this information to sketch the function. You could use your graphics calculator to check your sketch.

Strategy 2

To optimise a given situation

1. Express the dependent variable (say y) as a function of the independent variable (say x).
2. Differentiate y with respect to x
3. Let the derivative be zero and find the value of x that optimises the value of y
4. Examine the nature of the turning points to decide if it is a maximum or minimum.
5. Put your turning point in the context of the question.

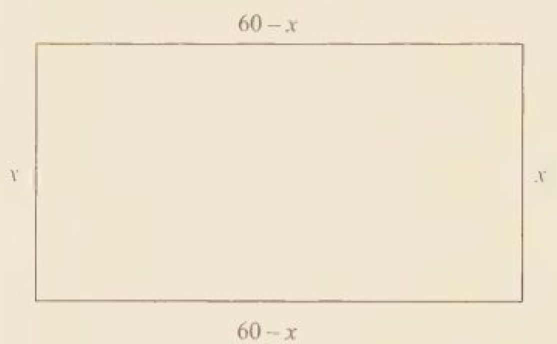
Example 4

A piece of rope 120 m long is to be used to draw out a rectangle on the ground.

What is the biggest area that can be enclosed in the rectangle?

Let x represent the height of the rectangle in metres,

So width $= \frac{1}{2}(120-2x) = 60-x$ — *Use the fact that perimeter = 2 × width + 2 × length = 120 to write width in terms of length.*



Identify your variables – a sketch helps.

The area, $A = x(60-x) = 60x - x^2$ — *(1) Express A, in terms of x*

$\frac{dA}{dx} = 60 - 2x$ — *(2) Differentiate with respect to x*

At a turning point,

$\frac{dA}{dx} = 0$ so $60 - 2x = 0$

$x = 30$ — *(3) Let the derivative be zero.*

Differentiating again gives

$\frac{d^2A}{dx^2} = -2$ — *(4) Find the nature of the turning point.*

This is less than zero, so the turning point is a maximum.

The greatest area will be enclosed when $x = 30$

So $A_{max} = 30(60-30)$

$= 900$

The greatest area that can be enclosed is $900\,m^2$. — *(5) Use the value of x to answer the question in context.*

Exercise 4.5B Reasoning and problem-solving

PURE

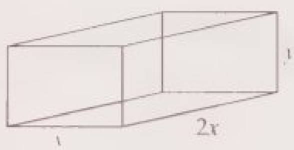
1 Two numbers, x and $1000-x$, add to make 1000. What would they need to be to maximise their product?

2 The product of two positive whole numbers, x and $\frac{3600}{x}$, is 3600. Their sum is the smallest it can be. What are the two numbers?

3 Two numbers, x and y, have a sum of 12

What values of x and y will make x^2y a maximum?

4 A wire model of a cuboid is made. The total length of wire used is 600 cm.



a Express y in terms of x

b Express the volume of the cuboid in terms of x

c Find the values of x and y which maximise the volume.

5 A rope of length 16 m is used to form three sides of a rectangular pen against a wall.

a How should the rope be arranged to maximise the area enclosed?

b Another length of rope is used to enclose an area of $50\,m^2$ in a similar way. What is the shortest length of rope that is needed?

6 The point $A(x, y)$ lies in the first quadrant on the line with equation $y = 6 - 5x$

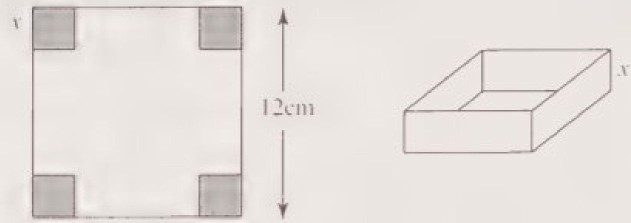
A rectangle with two sides on the coordinate axes has A as one vertex.

a Work out an expression in x for the area of the rectangle.

b Work out the point for which this area is a maximum.

7 A small tray is made from a 12 cm square of metal.

Small squares are cut from each corner and the edges left are turned up.



What should the side of the small square be so as to maximise the volume of the tray?

8 A fruit drink container is a cuboid with a square base. It has to hold 1000 ml of juice. Let one side of the square base be x cm and the height of the container be h cm.

a Express h in terms of x

b Find an expression for the surface area of the cuboid in terms of x

c Find the value of x that will minimise the surface area (and hence the cost of the container).

Challenge

9 A cylindrical container has to hold 440 ml of juice.

a Express its height h cm in terms of the radius x cm.

b The surface area of a cylinder with a lid and a base is given by

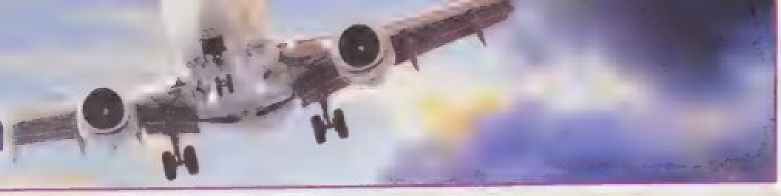
$$2\pi x^2 + 2\pi xh$$

Calculate the value of x that minimises this surface area.

Show your working and write your answer correct to 2 decimal places.

c If the container doesn't need a lid, how does this affect the answer?

4.6 Integration



Fluency and skills

The reverse process of differentiation is known is **integration**.

To differentiate sums of terms of the form ax^n you multiply by the power then reduce the power by 1 to get nax^{n-1}

To **integrate** you do the exact opposite: you add 1 to the power then divide by the new power.

When you differentiate a constant, the result is zero. So when you perform an integration, you should add a constant, c, to allow for this. This is referred to as the **constant of integration**. The value of this constant can only be determined if further information is given.

When you perform an integration you can check your result by differentiating it – you should get back to what you started with.

Key point

Integrating x^n with respect to x is written as

$$\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$$

When a function is multiplied by a constant, the constant can be moved outside the integral.

Key point

$\int a\mathrm{f}(x)\mathrm{d}x = a\int \mathrm{f}(x)\mathrm{d}x$ where a is a constant.

When integrating the sum of two functions, each function can be integrated separately.

You may see this rule referred to as the *sum rule*.
$\int(\mathrm{f}(x)+\mathrm{g}(x))\mathrm{d}x = \int \mathrm{f}(x)\mathrm{d}x + \int \mathrm{g}(x)\mathrm{d}x$

Given that displacement is a function of time $\mathrm{r}(t)$, then velocity $\mathrm{v}(t) = \mathrm{r}'(t)$. Reversing this, we get

Key point

$$\int \mathrm{v}(t)\mathrm{d}t = \mathrm{r}(t) + c$$

Given that velocity is a function of time $\mathrm{v}(t)$, then acceleration $\mathrm{a}(t) = \mathrm{v}'(t)$. Reversing this we get

Key point

$$\int \mathrm{a}(t)\mathrm{d}t = \mathrm{v}(t) + c$$

The Fundamental Theorem of Calculus shows how integrals and derivatives are linked to one another. The theorem states that, if $\mathrm{f}(x)$ is a continuous function on the interval $a \leq x \leq b$, then

Key point

$$\int_a^b \mathrm{f}(x)\mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a) \text{ where } \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{F}(x)) = \mathrm{f}(x)$$

Example 1

$f'(x) = 6x^3 + 3x^2 + \frac{1}{\sqrt{x}} + 4$

a Integrate to find f(x)

b Given that the point (1, 4) lies on the curve $y = f(x)$, find the constant of integration.

a $f(x) = \int\left(6x^3 + 3x^2 + \frac{1}{\sqrt{x}} + 4\right)dx$

$= \int 6x^3\,dx + \int 3x^2\,dx + \int x^{-\frac{1}{2}}\,dx + \int 4x^0\,dx$ — Express all the terms in index form and use the sum rule to isolate functions.

$= \frac{6x^4}{4} + \frac{3x^3}{3} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{4x^1}{1} + c$ — Integrate using $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$

$= \frac{3}{2}x^4 + x^3 + 2x^{\frac{1}{2}} + 4x + c$ — Simplify.

b $y = \frac{3}{2}x^4 + x^3 + 2x^{\frac{1}{2}} + 4x + c$

$4 = \frac{3}{2} + 1 + 2 + 4 + c$ — Substitute in coordinate values for *x* and *y*

$c = -\frac{9}{2}$ — Rearrange and evaluate.

Exercise 4.6A Fluency and skills

1 Work out the integral of each function with respect to *x*, remembering the constant of integration.

a 10 **b** $3x$

c $6x^2$ **d** $12x^3$

e $25x^4$ **f** x^6

g $3x+1$ **h** $5-4x$

i $3x^2+6x+2$ **j** $12x^2+6$

k $3-4x-6x^2$ **l** x^3+x^2+x+1

m $3-x-24x^3$ **n** $2x^3+4x+1$

o $x^{-4}+3x^{-2}$ **p** $2x^{-1}-x^{-2}$

q $x^{\frac{1}{2}}-x^3+1$ **r** $x^{\frac{1}{2}}-x^{-\frac{1}{2}}+x+6$

s $3x^{\frac{1}{3}}-x^{\frac{2}{3}}$ **t** $x^{\frac{1}{4}}-x^{\frac{3}{4}}+x^{-\frac{1}{2}}$

u $1-x^{-2}-x^{-\frac{2}{3}}$ **v** $\frac{1}{3}x^{-\frac{2}{3}}-x^4+4$

w $\frac{4}{3}\pi x^3-\frac{1}{3}\pi x^{\frac{1}{3}}+2\pi x$ **x** $x^2-x-1-x^{-2}$

2 Find

a $\int 1\,dx$ **b** $\int x\,dx$

c $\int(6x+7)\,dx$ **d** $\int(3-2x)\,dx$

e $\int(x^2+x+1)\,dx$ **f** $\int(1-4x-3x^2)\,dx$

g $\int(4x^3+2x-7)\,dx$ **h** $\int(2+9x^2-12x^3)\,dx$

i $\int\sqrt{x}\,dx$ **j** $\int\sqrt[3]{x}\,dx$

k $\int\sqrt{\frac{1}{x}}\,dx$ **l** $\int x^{\frac{2}{3}}\,dx$

3 Work out

a $\int\pi\,dx$ **b** $\int(3\pi+x)\,dx$

c $\int(x^2\sin 30^\circ)\,dx$ **d** $\int(x^2+6x)\,dx$

e $\int 4x^2+4x-28\,dx$ **f** $\int\left(x^2+\frac{1}{x^2}\right)dx$

g $\int\left(\sqrt{x}-\frac{2}{\sqrt{x}}\right)dx$ **h** $\int\left(8x-\frac{3}{x^2}\right)dx$

i $\int\left(\frac{1}{x^3}-\frac{1}{x^4}\right)dx$ **j** $\int\left(\frac{1}{x^2}-x-\frac{1}{x^3}\right)dx$

k $\int x+\frac{1}{\sqrt{x}}\,dx$ **l** $\int\left(x^2-\frac{3}{\sqrt{x}}+1\right)dx$

m $\int\left(\frac{1}{x^2}+\frac{3}{x^3}\right)dx$ **n** $\int x\sin\frac{\pi}{3}+x^2\sin\frac{\pi}{6}\,dx$

o $\int 3x^2-\frac{1}{\sqrt{x^3}}\,dx$ **p** $\int\left(\frac{x^2-x}{x}\right)dx$

q $\int\frac{x}{\sqrt{x}}\,dx$ **r** $\int\frac{x+1}{\sqrt{x}}\,dx$

4 The derivative and a point on the curve $y = \mathrm{f}(x)$ are given.

Work out the function $\mathrm{f}(x)$

a $\mathrm{f}'(x) = 6x^2$ at $(1, 5)$

b $\mathrm{f}'(x) = 12x^3$ at $(2, 18)$

c $\mathrm{f}'(x) = 5\sqrt{x}$ at $(9, 100)$

5 Work out the function, $\mathrm{f}(x)$ for the given $\mathrm{f}'(x)$ and the point $(x, \mathrm{f}(x))$

a $\mathrm{f}'(x) = 4x + 3$; $(2, 4)$

b $\mathrm{f}'(x) = 10$; $(1, 12)$

c $\mathrm{f}'(x) = 3x^2 + 2x + 1$; $(-2, 1)$

d $\mathrm{f}'(x) = 4x + 3\sqrt{x}$; $(1, 5)$

Reasoning and problem-solving

Strategy

To solve problems that require integration

1. Identify the variables and express the problem as a mathematical equation.
2. Integrate.
3. Use initial conditions to work out the constant of integration.
4. Substitute c into the equation and answer the question.

Example 2

A moving particle has acceleration $-10\ \mathrm{cms}^{-1}$

The particle starts from rest 50 cm to the right of the origin.

'The particle starts from rest' means when $t = 0$, $v = 0$

a Express the velocity as a function of time.

b Express the displacement from the origin as a function of time.

c State the acceleration, velocity and displacement after 3 seconds.

a $v(t) = \int a(t)dt$

$= \int -10 dt$

$= -10 \int 1 dt$

$= -10t + c$

$v(0) = -10 \times 0 + c = 0$

$c = 0$

So $v(t) = -10t$

b $s(t) = \int v(t)dt$

$= \int (-10t)dt$

$= -5t^2 + c$

$s(0) = -5 \times 0^2 + c = 50$

$c = 50$

So $s(t) = -5t^2 + 50$

c $a(3) = -10\ \mathrm{cm\,s^{-2}}$

$v(3) = -10 \times 3$

$= -30\ \mathrm{cm\,s^{-1}}$

$s(3) = -5 \times 3^2 + 50$

$= 5\ \mathrm{cm}$

1 Identify the variables and express the rate of change as a mathematical equation.

2 Integrate to work out the general form of the function.

3 Use the initial conditions to work out c

4 Use the value of c to answer the question.

Exercise 4.6B Reasoning and problem-solving

1 Given that the rate of change of P with respect to t is 7, and that when $t=4$, $P=2$

a Express P in terms of t

b Work out P when $t=5$

c Work out t when $P=16$

2 It is known that $f'(x)=1-6x$ and that $f(3)=6$

a Calculate $f(-2)$

b For what values of x does $f(x)=0$?

3 Kepler was an astronomer who studied the relationship between the orbital period, in y years, of a celestial body and the mean distance from the Sun, R (measured in astronomical units AU). The rate at which the period increases as the distance from the Sun increases is given by $\frac{dy}{dr}=\frac{3}{2}r^{\frac{1}{2}}$

a Express y in terms of r and c, the constant of integration.

b The Earth is 1 astronomical unit from the Sun and takes one year to orbit it. Find the relation between y and r

c Mars is 1.5 AU from the Sun. How long does it take to orbit the Sun?

d Saturn takes 29.4 years to orbit the Sun. What is its mean distance form the Sun?

4 The rate at which the depth, h metres, in a reservoir drops as time passes is given by $\frac{dh}{dt}=\frac{8}{3}t^3-4t, t>0$ where t is the time in days. When $t=0$, $h=4$

a Express h as a function of t

b What is the depth after half a day?

c When is the depth 16 m?

5 The second derivative of a function is given by $\frac{d^2y}{dx^2}=12x+1$

When $x=1$, $\frac{dy}{dx}=4$

a Work out an expression in x for $\frac{dy}{dx}$

b When $x=1$, $y=\frac{1}{2}$. What is the value of y when $x=2$?

6 The second derivative of a function is given by $\frac{d^2y}{dx^2}=6$

When $x=2$, $y=1$ and $\frac{dy}{dx}=3$

What is the value of y when $x=4$?

7 a A particle moves on the x-axis so that its acceleration is a function of time, $a(t)=2t$. Initially (at $t=0$) the particle was 2 cm to the right of the origin travelling with a velocity, v, of 2 cm s^{-1}.

i Express the velocity and the displacement, s, as a function of t

ii Calculate both velocity and displacement when $t=1$

b Repeat part **a** for particles with the following initial conditions.

i $a(t)=6$; $s(0)=-2$; $v(0)=5$

ii $a(t)=t+1$; $s(0)=1$; $v(0)=2$

iii $a(t)=t^2$; $s(0)=0$; $v(0)=-1$

8 A piece of rock breaks away from the White Cliffs of Dover.

Its acceleration towards the shore, 90 m below, is 5 m s^{-2}.

Let t seconds be the time since the rock broke free. Let $v(t)$ m s^{-1} be its velocity at time t. Note that when $t=0$, the velocity of the rock is 0

How long will it take for the rock to hit the shore?

Challenge

9 The velocity of a particle is given by $v(t)=4+3t$ where distance is measured in metres and time in seconds. After one second the particle is 6 m to the right of the origin.

a Where was the particle initially?

b What is its acceleration at $t=5$?

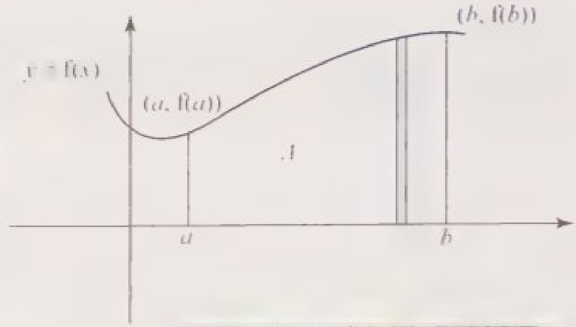
c How far has it travelled in the fifth second?

d Work out an expression in n for the distance travelled in the n^{th} second.

4.7 Area under a curve

Fluency and skills

You can use integration to find the area between a curve and the x-axis. To do this, you perform a calculation using a **definite integral**.



Key point

A definite integral is denoted by $\int_a^b f(x)\,dx$

b is called the **upper limit**, and a the **lower limit**.

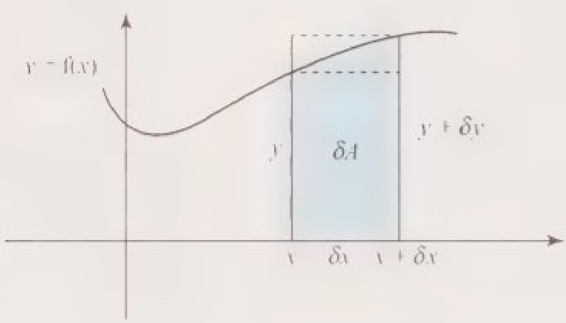
Consider a continuous function $y = f(x)$ over some interval and where all points on the curve in that interval lie on the same side of the x-axis. The area, A, is bound by $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ where $a < b$

Consider a small change in x and the change in area, δA, that results from this change.

Use Leibniz notation where δx represents a small change in x and δy represents the corresponding change in y

ICT Resource online

To experiment with numerical integration using rectangles, click this link in the digital book.



Zooming in you can see a small section of the area, trapped between vertical lines at x and at $x + \delta x$

The small area between the vertical lines at x and $x + \delta x$ (shaded) is denoted by δA

$$y\delta x \le \delta A \le (y + \delta y)\delta x$$

Dividing by δx: $$y \le \frac{\delta A}{\delta x} \le (y + \delta y)$$

As you let δx tend to zero:

$$y \le \lim_{\delta x \to 0} \frac{\delta A}{\delta x} \le \lim_{\delta x \to 0}(y + \delta y) \Rightarrow y \le \frac{dA}{dx} \le y$$

Remember

$\lim_{\delta x \to 0} \delta y = 0$ and $\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$

So $$y = \frac{dA}{dx}$$

Integrating this will give you a formula for the area from the origin up to the upper bound, namely $A = \int y\,dx$. Note that, when calculating this small area, the lines $x = a$ and $x = b$ were not used so this has given us a *general* formula for calculating the area. A further calculation is needed to obtain the area between $x = a$ and $x = b$

PURE

If you wish to calculate the area between two vertical lines, $x = a$ and $x = b$, then you need only integrate to get the formula for area and substitute a and b for x. The difference between your results is the required area.

If $\int f(x)dx = F(x) + c$ then $(F(b) + c) - (F(a) + c) = F(b) - F(a)$

The constants of integration sum to zero. This means you don't need to worry about the constant of integration when calculating a definite integral.

Key point

The area under a curve between the x-axis, $x = a$, $x = b$ and $y = f(x)$, is given by $A = \int_a^b f(x)dx = F(b) - F(a)$

Working with areas below the x-axis will produce negative results. As area is a positive quantity you should use the magnitude of the answer only (ignore the negative sign).

Example 1

Evaluate the definite integral $\int_1^4 (3x^2 + 4x + 1)dx$. You must show your working.

$\int_1^4 (3x^2 + 4x + 1)dx$

$= [x^3 + 2x^2 + x]_1^4$ — Integrate

$= (4^3 + 2\times4^2 + 4) - (1^3 + 2\times1^2 + 1)$ — Substitute the two values of x

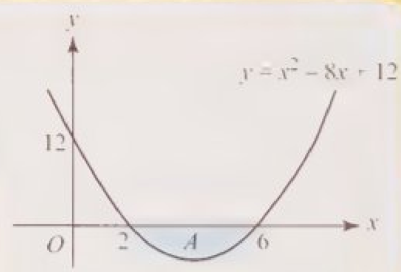
$= 100 - 4$

$= 96$

Example 2

A parabola cuts the axes as shown.

Show that the area A is $10\frac{2}{3}$ units.



$A = \int_2^6 x^2 - 8x + 12dx$

$= \left[\frac{x^3}{3} - \frac{8x^2}{2} + 12x\right]_2^6$ — Integrate.

$= \left(\frac{216}{3} - 4\times36 + 12\times6\right) - \left(\frac{8}{3} - 4\times4 + 12\times2\right)$ — Substitute the two values of x

$= -10\frac{2}{3}$

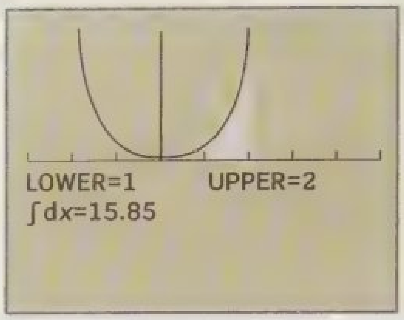
Area A is $10\frac{2}{3}$ units^2. — Remember, areas must always be positive, so you can ignore the minus sign.

If you had calculated the definite integral between 0 and 6, the answer would be zero. The positive and negative sections would cancel each other out.

Calculator

Try it on your calculator

You can use a calculator to evaluate a definite integral.



Activity

Find out how to evaluate $\int_1^2 3x^4 - x^3 + 1\,dx$ on *your* calculator.

Exercise 4.7A Fluency and skills

1 Evaluate these definite integrals. Show your working in each case.

a $\int_2^4 x+1\,dx$ **b** $\int_1^5 2x-1\,dx$

c $\int_0^3 4-x\,dx$ **d** $\int_{-2}^6 1-3x\,dx$

e $\int_{-2}^3 7\,dx$ **f** $\int_1^7 \pi\,dx$

g $\int_1^3 \pi+1\,dx$ **h** $\int_0^3 3x^2+4x+1\,dx$

i $\int_{-3}^3 x^2-6x-1\,dx$ **j** $\int_{-1}^1 1-x-x^2\,dx$

k $\int_{\sqrt{2}}^{\sqrt{3}} x^3\,dx$ **l** $\int_1^{1.5} x+\frac{1}{x^2}\,dx$

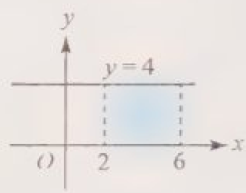
m $\int_{-1}^0 x^3+2x+3\,dx$ **n** $\int_4^{25} \frac{1}{\sqrt{x}}\,dx$

o $\int_2^5 4\pi x^2\,dx$ **p** $\int_{\sqrt{3}}^{\sqrt{5}} x^3+x\,dx$

q $\int_{-3}^3 x^3-2x\,dx$ **r** $\int_{\frac{1}{4}}^{\frac{1}{2}} \frac{\pi}{x^2}\,dx$

2 Work out the total shaded area for each of these graphs. Show your working in each case.

a



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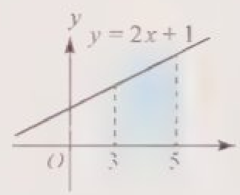

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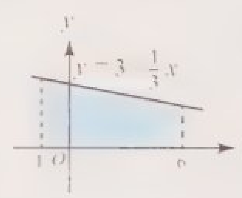

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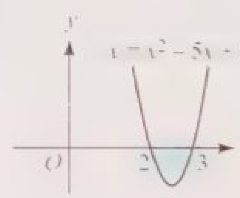

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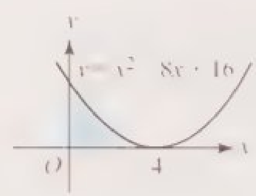

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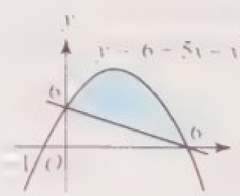

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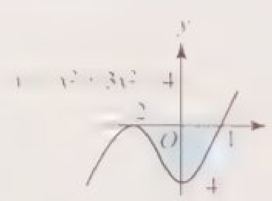

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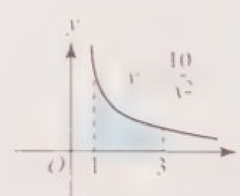
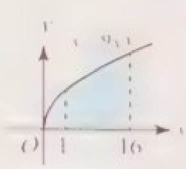


i



Reasoning and problem-solving

Strategy

To calculate the area under a curve

1. Make a sketch of the function, if there isn't one provided.
2. Identify the area that has to be calculated.
3. Write down the definite integral associated with the area.
4. Evaluate the definite integral and remember that the area is always positive.

Exercise 4.7B Reasoning and problem-solving

1 An area of 29 units² is bound by the line $y=x$, the x-axis and the lines $x=-3$ and $x=a$, $a<0$. Use integration to help you calculate the value of a

2 Integrate the function $\mathrm{f}(x)=x^2-6x+8$ with respect to x and thus calculate the area bound by the curve $y=\mathrm{f}(x)$, the x-axis, and the lines $x=2$ and $x=4$

3 A parabola has equation $y=10+3x-x^2$

Calculate the area of the dome trapped between the curve and the x-axis.

4 The equation $y=\sqrt{x}$ is one part of a parabola with $y=0$ as the axis of symmetry.

a Calculate $\int y\,\mathrm{d}x$

b Calculate the area trapped between the curve, the x-axis and the lines $x=0$ and $x=1$

c P is the area below the curve between $x=1$ and $x=4$. Q is the area between $x=4$ and $x=9$. Calculate the ratio P : Q

d Calculate the area trapped between the curve $y=1+\sqrt{x}$, the x-axis and the lines $x=0$ and $x=1$ and explain the difference between this and the answer to part **b**.

5 **a** Area A is defined by $\mathrm{A}(a)=\int_1^a 3x^2\,\mathrm{d}x$

For what value of a is the area 999 units²?

b Area B is defined by $\mathrm{B}(a)=\int_a^{2a} 3x^2\,\mathrm{d}x$

For what value of a is the area 875 units²?

c Area C is defined by $\mathrm{C}(a)=\int_a^{a+1} 3x^2\,\mathrm{d}x$

For what values of a is the area 19 units²?

6 The area trapped between the x-axis and the parabola $y=(x+3)(x-9)$ is split in two parts by the y-axis. What is the ratio of the larger part to the smaller?

7 An area is bounded by the curve $y=4x^3$, the x-axis, the line $x=1$ and the line $x=a$, $a>1$. What is the value of a if the area is 2400 units²?

8 Sketch a graph of $y=\sin x$ for $0\le x\le 360°$ and, using this, explain why the definite integral $\int_0^{360}\sin x°\,\mathrm{d}x$ equals zero.

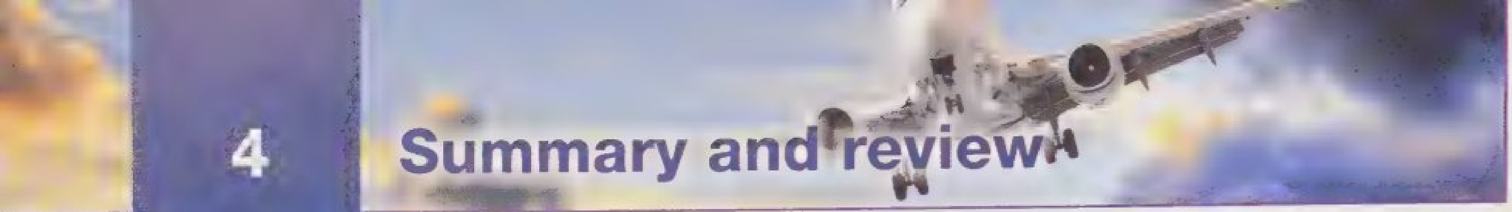
9 On a velocity–time curve, the distance travelled can be obtained by calculating the area under the curve.

An object is thrown straight up. Its velocity, in m s⁻¹, after t seconds can be calculated using $\mathrm{v}(t)=24t-5t^2$

What distance did the object travel between $t=1$ and $t=4$?

Challenge

10 Space debris is detected falling into the Earth's atmosphere. Its velocity in kilometres per second is modelled by $\mathrm{v}(t)=5+0.01t$ where t is the time in seconds measured from where the debris was detected. It completely burned up after 4 seconds. How far did the debris travel in the atmosphere?



4 Summary and review

Chapter summary

- The gradient of the tangent to a curve at the point P can be approximated by the gradient of the chord PQ, where Q is a point close to P on the curve.
- The derivative at the point P on the curve can be calculated from first principles by considering the limiting value of the gradient of the chord PQ as Q tends to P
- The derivative is denoted by $f'(x)$

  $$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

- If $y=f(x)$ then $\frac{dy}{dx}=f'(x)$
- The derivative gives the instantaneous rate of change of y with respect to x.
- The derivative of $y=ax^n$, where a is a constant, is $\frac{dy}{dx}=nax^{n-1}$
- $f(x)=ag(x)+bh(x)\Rightarrow f'(x)=ag'(x)+bh'(x)$, where a, b are constants.
- $f'(x)$ gives the rate of change of function f with respect to x. So,

  $f'(x)>0$ means the function is increasing,
  $f'(x)<0$ means the function is decreasing,
  $f'(x)=0$ means the function is stationary.

- By sketching the gradient function, you can determine whether a function is increasing, decreasing, or stationary.
- At a turning point, the tangent is horizontal, so $f'(x)=0$
- Where the function changes from being an increasing to a decreasing function, the turning point is referred to as a maximum.
- Where the function changes from being a decreasing to an increasing function, the turning point is referred to as a minimum.
- The derivative is itself a function that can be differentiated. The result is called the second derivative and is denoted by $f''(x)$ or $\frac{d^2y}{dx^2}$
- $\frac{d^2y}{dx^2}$ gives the rate of change of the gradient with respect to x. Assuming $\frac{dy}{dx}=0$, then $f''(x)>0$ means the gradient is increasing—it is a minimum turning point.

  $f''(x)<0$ means the gradient is decreasing—it is a maximum turning point.
- Since the derivative at a point $(a, f(a))$ gives the gradient of the curve at that point and the gradient of the tangent at that point then $m_{tangent}=f'(a)$ and the equation of the tangent is $y-f(a)=f'(a)(x-a)$
- The normal at $(a,f(a))$ has a gradient $m_{normal}=-\frac{1}{f'(a)}$ so the equation of the normal is $y-f(a)=-\frac{1}{f'(a)}(x-a)$

- The process of differentiation can be reversed by a process called integration.

  $F'(x) = f(x) \Rightarrow F(x) = \int f(x)\,dx + c$

  where c is the constant of integration.

- $\int ax^n\,dx = \frac{ax^{n+1}}{n+1}$ ... read as "the integral of ax^n with respect to x"
- $\int (af(x) + bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx$
- $\int_a^b f(x)\,dx$ is called a definite integral with upper limit b and lower limit a
- $\int f(x)\,dx = F(x) + c \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$
- If $f(x)$ is continuous in the interval $a \le x \le b$ and all points on the curve in this interval are on the same side of the x-axis, the area bounded by the curve, the x-axis and the lines $x = a$ and $x = b$ is given by the positive value of $\int_a^b f(x)\,dx$

Check and review

You should now be able to...	Try Questions
✔ Differentiate from first principles.	1
✔ Differentiate functions composed of terms of the form ax^n	2, 3
✔ Use differentiation to calculate rates of change.	4
✔ Work out equations, tangents and normals.	5
✔ Work out turning points and determine their nature.	6
✔ Work out and interpret the second derivative.	7
✔ Work out the integral of a function.	8
✔ Understand and calculate definite integrals.	9
✔ Use definite integrals to calculate the area under a curve.	10

1 Differentiate each function from first principles.

a $y=3x^2$ at $(1,3)$ b $y=x^3+1$ when $x=-1$ c $y=x^4-x-5$ at $(2,9)$

d $y=1-x^2$ e $y=x-x^2$ f $y=\pi x^2$

2 Work out the derivative of

a $f(x)=x^3+2x^2+3x+1$ b $y=4\sqrt{x}+x$ c $y=x+\frac{1}{x}+\frac{1}{x^2}$

d $f(x)=\sqrt[4]{x}-\sqrt[3]{x}$ e $y=\frac{x+3}{x^2}$ f $y=\frac{4}{x}+\frac{2}{\sqrt{x}}$

g $y=1-\frac{1}{x^2}+\frac{1}{\sqrt[3]{x}}$ h $y=\frac{x^2+2x+3}{x}$ i $y=\frac{x+2\sqrt{x}}{\sqrt{x}}$

j $y=\frac{\sqrt{x}+1}{\sqrt[3]{x}}$

3 Work out the value of the following functions. Show your working.

a $f'(2)$ when $f(x)=2x^2-5$ b $\frac{dy}{dx}$ when $x=3$ given $y=\frac{x+3}{x}$

c $f'(4)$ when $f(x)=1+\frac{1}{x}-\frac{33}{\sqrt{x}}$ d $\frac{dy}{dx}$ when $x=9$ given $y=\frac{x-9}{2\sqrt{x}}$

4 a Work out the rate of change of y with respect to x when $x=4$ given that $y=x(2x^2-5x)$
Show your working.

b A roll of paper is being unravelled. The volume of the roll is a function of the changing radius, r cm. $V=25\pi r^2$ [V cm^3 is the volume].

Calculate the rate of change of the volume when the radius is 3 cm. Show your working.

c A particle moves along the x-axis so that its distance, D cm, from the origin at time t seconds is given by $D(t)=t^2-5t+1$

i Work out the particle's velocity at $t=3$ (the rate of change of distance with respect to time). Show your working.

ii Work out the particle's acceleration at this time (the rate of change of velocity with time). Show your working.

d A tourist ascends the outside of a tall building in a scenic elevator.

As she ascends, she can see further. The distance to her horizon, K km, can be calculated by the formula $K=\sqrt{\frac{hD}{1000}}$ where h is her height in metres and D is the diameter of the planet in kilometres. On Earth, $D=12\,742$ km

i Calculate the rate of change of the distance to her horizon with respect to her height
1 When she is 50 m up,
2 When she is 100 m up.

ii How high up will she be when the distance to the horizon is changing at a rate of 0.4 km per m of height?

iii Suppose the tourist were on a building on the Moon. The Moon has a diameter of 3474 km. Calculate the rate of change of the distance to her horizon with respect to her height when she is 50 m up.

5 Work out the equations of the tangent and normal to the given curves at the given points. Show your working.

a $y=x^2+4x+3$ at $(1,8)$

b $y=2x-\frac{3}{x}$ at $(3,5)$

c $y=5+4x-x^2$ at $(2,9)$

d $y=200\sqrt{x}$ at $(25,1000)$

6 Work out the turning points on each curve and determine their nature. Show your working.

a $y=x^2+4x-5$

b $y=3x^3-4x$

c $y=10x^4$

d $y=ax+\frac{a}{x}$ where a is a positive constant.

e $y=ax^2+bx+c$ where a, b and c are positive constants.

7 Work out the second derivative of

a $y=x^2$

b $y=x^3+x^2$

c $y=\frac{1}{x}$

d $y=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}$

8 Calculate

a $\int 1-x-x^3\,dx$

b $\int \frac{1}{x^2}-4x^2\,dx$

c $\int \sqrt{x}-\frac{1}{\sqrt{x}}-\sqrt[3]{x}\,dx$

d $\int \frac{x^2-1}{\sqrt{x}}\,dx$

9 Calculate the following definite integrals. Show your working.

a $\int_1^4 x+6\,dx$

b $\int_1^{10} 2\pi\,dx$

c $\int_{-2}^4 x^2-4x-5\,dx$

d $\int_{\sqrt{2}}^{\sqrt{5}} 2x^3\,dx$

e $\int_1^9 \frac{x+1}{2\sqrt{x}}\,dx$

10 a Calculate the area bounded by $y=x^2-7x+10$, the x-axis, $x=2$ and $x=5$. Show your working.

b Calculate the area bounded by $y=x^2-5x+6$, the x-axis, $x=2$ and $x=3$. Show your working.

c Calculate the area bounded by $y=27-x^3$, the x-axis, and the y-axis. Show your working.

What next?

Score		
0 – 5	Your knowledge of this topic is still developing. To improve, search in MyMaths for the codes: 2028–2030, 2054–2056, 2269–2270, 2273	
6 – 8	You're gaining a secure knowledge of this topic. To improve, look at the InvisiPen videos for Fluency and skills (04A)	
9 – 10	You've mastered these skills. Well done, you're ready to progress! To develop your techniques, look at the InvisiPen videos for Reasoning and problem-solving (04B)	

Click these links in the digital book

Exploration

Going beyond the exams

History

Differentiation and integration belong to a branch of maths called calculus, which is the study of change. It took hundreds of years and the work of many mathematicians to develop calculus to the state in which we know it today.

The first person to write a book about both differentiation and integration, and the first woman to write any book about mathematics, was **Gaetana Maria Agnesi**. One of 21 children, Maria displayed incredible ability in a number of disciplines at an early age, and was an early campaigner for women's right to be educated – giving a speech on the topic when she was 9 years old.

Agnesi's book, **Istituzioni analitiche ad uso della gioventu italiana** (analytical institutions for the use of the Italian youth), was published in 1748 when she was just 30 years old.

Research

Before Maria, in the 17th century, the fundamental theorem of calculus was independently discovered by **Isaac Newton** and **Gottfried Leibniz**.

These two were both well-regarded by the mathematical community, and both published very similar works on the subject within only a few years of each other. This led to suspicions of copying and a bitter feud between the two lasted until Leibniz's death in 1716.

Find out the contributions that Newton and Leibniz made to the theorem of calculus and use your research to write a summary. In your summary, discuss any similarities and differences between the works of each mathematician, and how their results added to or built on what was previously established in the field of calculus.

Isaak Newton.

Gottfried Wilhelm Leibniz.

1 $y=5x^3-2x^{-3}$ Calculate and select the correct answer for

a $\frac{dy}{dx}$

A $15x^2+6x^{-2}$ B $15x^2-6x^{-4}$ C $15x^2-6x^{-2}$ D $15x^2+6x^{-4}$ **[1 mark]**

b $\int y\,dx$

A $\frac{5x^4}{4}-x^{-2}+c$ B $\frac{5x^4}{4}+x^{-2}+c$ C $\frac{5x^4}{4}-\frac{x^{-4}}{2}+c$ D $\frac{5x^4}{4}+\frac{x^{-4}}{2}+c$ **[1]**

2 Calculate the exact value of $\int_1^2 5x-2\,dx$. Select the correct answer.

A 5.5 B 1.5 C 13 D 8 **[1]**

3 The radius (in cm) of a circle at time t seconds is given by $r=20-2\sqrt{t}$

a Work out an expression for the rate of change of the radius. **[3]**

b Calculate the rate of change of the radius at time 25 s. State the units of your answer. **[2]**

4 $y=2x^2-x^{-3}$

a Calculate the gradient of the curve at the point $(1,1)$ **[3]**

b Work out the equation of the normal to the curve at the point $(1,1)$ **[3]**

5 $f(x)=\frac{1}{5}x^5-x^2$

a Work out an expression for $f'(x)$ **[2]**

b Calculate $f'(2)$ **[2]**

c Work out the equation of the tangent to $y=f(x)$ at the point where $x=2$

Give your equation in the form $ax+by+c=0$ where a, b and c are integers. **[4]**

6 The curve C has equation $y=6x^3-3x^2-12x+5$

a Use calculus to show that C has a turning point when $x=-\frac{2}{3}$ **[4]**

b Work out the coordinates of the other turning point on C **[2]**

c Is this point is a maximum or a minimum? Explain your reasoning. **[2]**

7 Work out the range of values of x for which $y=x^3+5x^2-8x+4$ is decreasing. **[5]**

8 Work out these integrals.

a $\int 2x+3x^5\,dx$ **[2]** b $\int x^{-4}-4x^3\,dx$ **[3]** c $\int 3\sqrt{x}+\frac{1}{\sqrt{x}}\,dx$ **[3]**

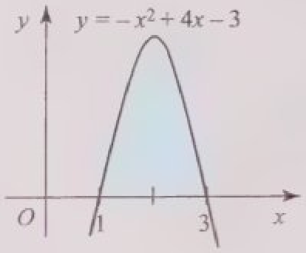
9 Calculate the exact values of these definite integrals. You must show your working.

a $\int_0^3 2\sqrt{x}\,dx$ **[4]** b $\int_1^2 \frac{2}{x^3}-3x\,dx$ **[4]**

10 Find an expression for $f(x)$ when

a $f'(x)=4x^2+5x-1$ **[3]** b $f'(x)=7x^{-3}-x+\sqrt{x}$ **[5]**

11 The region shown is bounded by the x-axis and the curve $y=-x^2+4x-3$



Show that the area of the shaded region is $1\frac{1}{3}$ square units. **[4]**

12 $y=3x^2$

a Work out $\frac{dy}{dx}$ from first principles. **[5]**

b Calculate the gradient of the tangent where $x=5$ **[2]**

13 $y=x^3-2x$

a Work out $\frac{dy}{dx}$ from first principles. **[5]**

b Calculate the gradient of the tangent where $x=2$ **[2]**

14 Work out $\frac{dy}{dx}$ when

a $y=x(x+3)^2$ **[4]** b $y=\frac{x+2}{\sqrt{x}}$ **[4]**

15 The volume (in litres) of water in a container at time t minutes is given by

$$V=\frac{t^3-8t}{3t}$$

Calculate the rate of change of the volume after 4 minutes. **[6]**

16 Work out the equation of the normal to $y=x^2(2x+1)(x-3)$ at the point where $x=2$ **[10]**

17 The curve C has equation $y=\frac{3x^3+\sqrt{x}}{2x}$

a Work out the equation of the tangent to C at the point where $x=1$ **[9]**

The tangent to C at $x=1$ crosses the x-axis at the point A and the y-axis at the point B

b Calculate the exact area of the triangle AOB **[4]**

18 Show that the function $f(x)=(1+2x)^3$ is increasing for all values of x **[6]**

19 Work out the range of values of x for which $f(x)=5\sqrt{x}+\frac{3}{\sqrt{x}}$ is a decreasing function. **[4]**

20 Work out the range of values of x for which $y=\frac{1}{3}x(x-1)(5-x)$ is an increasing function. **[7]**

21 Given that $f(x)=x^2(2x-\sqrt{x})$, work out expressions for

a $f'(x)$ **[4]** ii $f''(x)$ **[2]**

22 Given that $y=3x^2-4\sqrt{x}$, work out $\frac{d^2y}{dx^2}$ **[4]**

23 Calculate the coordinates of the stationary point on the curve with equation

$$y=32x+\frac{2}{x^2}-15,\ x>0$$

Show that this point is a minimum. **[8]**

24 A cylindrical tin is closed at both ends and has a volume of $200\ cm^3$.

a Express the height, h in terms of the radius, x **[3]**

b Show that the surface area, A of the tin is given by

$$A=2\pi x^2+\frac{400}{x}$$ **[3]**

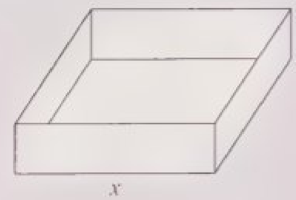
c Calculate the value of x for which A is a minimum. **[4]**

d Hence, work out the minimum value of A **[2]**

e Justify that the value found in part **d** is a minimum. **[2]**

25 A box has a square base of side length x

The volume of the box is $3000\,\text{cm}^3$.



a Show that the surface area, A, of the box (not including the lid) is given by $A = x^2 + \dfrac{12000}{x}$ **[6]**

b Calculate the value of x for which A is a minimum. **[3]**

c Hence, work out the minimum value of A **[2]**

d Justify that the value found in part **c** is a minimum. **[2]**

26 Work out these integrals.

a $\int (2x+3)^2\,dx$ **[3]** b $\int \sqrt{x}(5x-1)\,dx$ **[4]** c $\int \dfrac{2+x}{2\sqrt{x}}\,dx$ **[4]**

27 Calculate the exact values of these definite integrals. You must show your working.

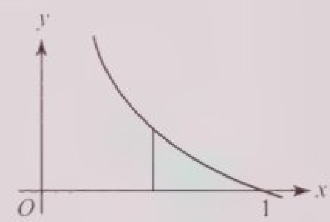
a $\int_0^1 (3x-1)^3\,dx$ **[5]** b $\int_{-1}^1 x^2(x-4)(x-5)\,dx$ **[5]**

28 Calculate the area of the region bounded by the x-axis and the curve

$y = x^2 - 3x - 10$ **[8]**

29 The shaded region shown is bounded by the x-axis, the line $x = \dfrac{1}{2}$ and the curve with equation $y = \dfrac{5}{x^2} - 3x^2 - 2\sqrt{x},\ x > 0$

Calculate the area of the shaded region. **[5]**



30 The curve with equation $y = f(x)$ passes through the point $(1, 1)$

Given that $f'(x) = 5x^4 - \dfrac{2}{x^3}$

a Calculate $f(x)$ **[4]**

b Work out the equation of the normal to the curve at the point $(1, 1)$ **[4]**

31 a Differentiate with respect to x, where k is a constant.

i $kx + x^k,\ k \neq -1$ ii $\dfrac{1}{x^k} - k,\ k \neq 1$ **[4]**

b Integrate the functions in part **a** with respect to x **[5]**

32 $f(x) = x^3 - 2x$

The tangent to $y = f(x)$ through the point where $x = 2$ meets the normal through the point $x = -1$ at the point P. Calculate the coordinates of P **[13]**

33 The function $f(x)$ is given by $f(x) = x^2 + kx$ where k is a positive constant.

The tangent to $y = f(x)$ at the point where $x = k$ meets the x-axis at the point A and the y-axis at the point B

Given that the area of the triangle ABO is 36 square units, work out the value of k **[10]**

34 The normal to the curve $y = 2x^2 - x + 2$ at the point where $x = 1$ intersects the curve again at the point Q. Calculate the coordinates of Q **[9]**

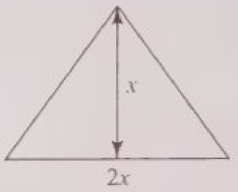
35 Work out and classify all the stationary points of the curve with equation $y=x^4-2x^2+1$ [8]

36 Work out and classify all the stationary points of the curve with equation $y=3x^4+4x^3-12x^2+20$ [8]

37 Calculate the range of values of x for which $f(x)=4x^4-2x^3$ is an increasing function. [5]

38 Calculate the range of values of x for which $f(x)=-3x^4+8x^3+90x^2+12$ is a decreasing function. [6]

39 A triangular prism has a cross-section with base twice its height.



The volume of the prism is 250 cm^3.

Calculate the minimum possible surface area of the prism given that it is closed at both ends. [12]

40 A cylinder of radius, r cm is open at one end. The surface area of the cylinder is 700 cm^2.

Calculate the maximum possible volume of the cylinder. [11]

41 The curve with equation $y=f(x)$ passes through the point $A(1,4)$

Given that $f'(x)=3x^2-\frac{2}{x^3}$

a Work out the equation of the tangent to the curve at the point when $x=-1$ [7]

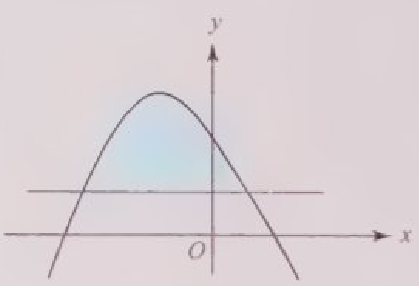
The tangent crosses the y-axis at the point B

b Calculate the area of the triangle ABO [3]

42 The curve with equation $y=f(x)$ passes through the point $(0,0)$

Given that $f'(x)=4x-3x^2$, work out the area enclosed by the curve $y=f(x)$ and the x-axis. [8]

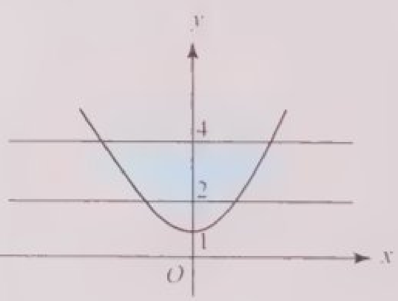
43 The shaded region is bounded by the curve with equation $y=12-7x-x^2$ and the line with equation $y=4$



Calculate the area of the shaded region. [9]

44 Calculate the area of the region bounded by the x-axis and the curve with equation $y=x(x+1)(x-2)$ [8]

45 The shaded region is bounded by the curve with equation $y=3x^2+1$ and the lines $y=4$ and $y=2$



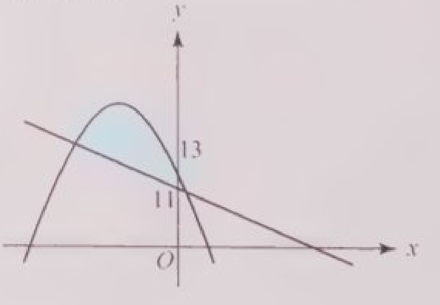
Calculate the area of the shaded region. [11]

46 The region R is bounded by the x-axis and the curve with equation

$y=x^2(k-x)$, where k is a positive constant.

Given that the area of R is 108 square units, calculate the value of k [7]

47 The region R is bounded by the curve with equation $y=13-2x-x^2$ and the line $y=11-x$



Calculate the area of R [9]