3 Trigonometry

GPS uses a technique called trilateration to calculate positions. The receiver, a mobile phone for example, receives direct signals from four different satellites simultaneously. The imaginary lines between the satellites and the receiver form the sides of triangles, which are then used by the mobile to calculate its position. Trilateration is a hi-tech version of triangulation, a technique that requires the use of trigonometry.

Trigonometry is the study of the relationships between angles and the sides of a triangle. It is immensely useful in fields such as astronomy, engineering, architecture, geography and navigation, as it allows you to calculate distances and angles or bearings. The sine and cosine functions are periodic in nature. This makes them highly useful in modelling periodic phenomena, and they can be used to describe different types of wave, including sound and light waves.

Orientation

What you need to know

KS4
- Apply and derive Pythagoras' theorem.
- Recognise graphs of trigonometric functions.
- Apply some properties of angles and sides of a triangle.

Ch1 Algebra 1 p.12
- Working with surds.

What you will learn

- To calculate the values of sine, cosine and tangent for any angle.
- To use trigonometric identities and recognise the equation of a circle.
- To sketch and describe trigonometric functions.
- To solve trigonometric equations.
- To use the sine and cosine rule and the area formula for a triangle.

What this leads to

Ch14 Trigonometric identities
Radians.
Reciprocal and inverse trigonometric functions.
Compound angles.
Equivalent forms for $a\cos\theta + b\sin\theta$

Ch16 Integration and differential equations
Integrating trigonometric functions.

MyMaths Practise before you start: 1112, 1131, 1133, 2024, 2036

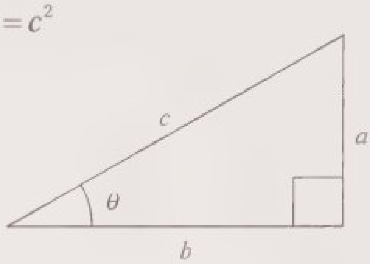
3.1 Sine, cosine and tangent

Fluency and skills

You can use trigonometry to find lengths and angles in right-angled triangles. This branch of mathematics is used in engineering, technology and many sciences.

Pythagoras' theorem for right-angled triangles is $a^2+b^2=c^2$

Dividing by c^2 gives $\frac{a^2}{c^2}+\frac{b^2}{c^2}=\frac{c^2}{c^2}$ or $\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1$



$\sin\theta=\frac{a}{c}$

$\cos\theta=\frac{b}{c}$

$\tan\theta=\frac{a}{b}$

See p.300 For a list of mathematical notation.

$\sin^2\theta+\cos^2\theta\equiv1$ **Key point**

$\tan\theta=\frac{a}{b}$

Dividing numerator and denominator of $\tan\theta$ by c gives a definition for $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$

$\tan\theta\equiv\frac{\frac{a}{c}}{\frac{b}{c}}\equiv\frac{\sin\theta}{\cos\theta}$ **Key point**

These two identities are true for all values of θ

Example 1

Calculate **a** $\sin\theta$ **b** $\tan\theta$ as surds, given that θ is acute and $\cos\theta=\frac{1}{\sqrt{3}}$

a $\sin^2\theta=1-\cos^2\theta=1-\left(\frac{1}{\sqrt{3}}\right)^2=\frac{2}{3}$

$\sin\theta=\sqrt{\frac{2}{3}}$ — θ is acute, so ignore $-\sqrt{\frac{2}{3}}$

b $\tan\theta=\frac{\sin\theta}{\cos\theta}$ — Use the identities.

$=\sqrt{\frac{2}{3}}\times\frac{\sqrt{3}}{1}=\sqrt{\frac{2\times3}{3\times1}}$ — Substitute values and simplify.

$\tan\theta=\sqrt{2}$

See Ch1.3 For a reminder of simplifying surds.

Example 2

Prove that $\tan\theta+\frac{1}{\tan\theta}\equiv\frac{1}{\sin\theta\cos\theta}$

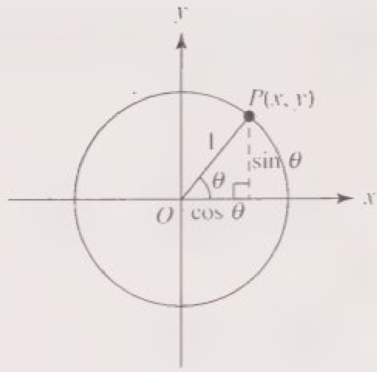
$\tan\theta+\frac{1}{\tan\theta}\equiv\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}$ — Use $\tan\theta=\frac{\sin\theta}{\cos\theta}$

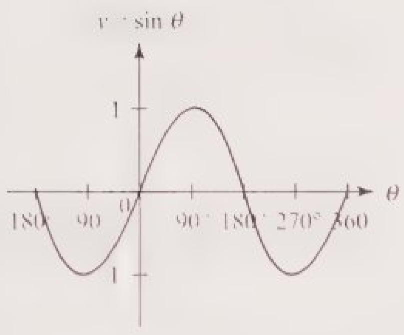
$\equiv\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$ — Add the two fractions.

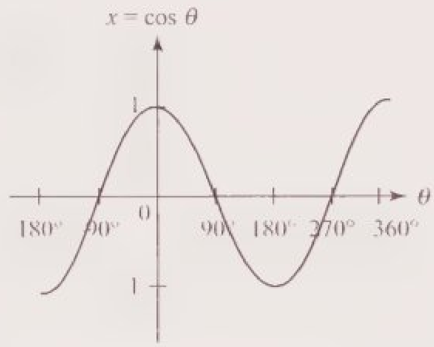
$\equiv\frac{1}{\sin\theta\cos\theta}$ — Use $\sin^2\theta+\cos^2\theta=1$

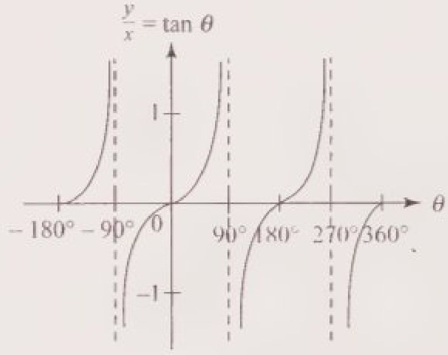
You can use the unit circle to draw graphs of the trigonometric ratios. The point P moving around the circle, centre O, has coordinates $x = \cos\theta$ and $y = \sin\theta$

As $P(x, y)$ moves around the circle you can plot graphs of the values of y, x and $\frac{y}{x}$ for each value of θ









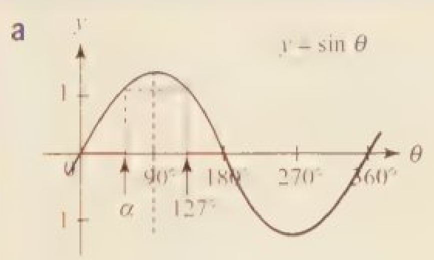
Extending the graphs for higher and lower values of θ shows they are all **periodic functions**, with a period of 360° for sine and cosine and 180° for tangent.

You can also see the symmetries from the graphs. For example, $y = \sin\theta$ has lines of symmetry at $\theta = -90°$, $\theta = 90°$, $\theta = 270°$, ... and it has rotational symmetry (order 2) about every point where the graph intersects the θ-axis.

Example

Express **a** $\sin 127°$ **b** $\cos 132°$ in terms of acute angles.

a



$\sin 127°$ is positive.

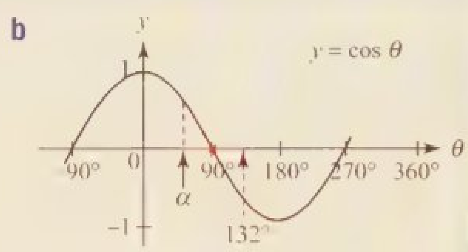
$127° = 180° - 53°$

Use the line of symmetry $\theta = 90°$

$\alpha = 53°$

$\sin 127° = \sin 53°$

b



$\cos 132°$ is negative.

$132° - 90° = 42°$

Use rotational symmetry about (90°, 0)

$\alpha = 90° - 42° = 48°$

$\cos 132° = -\cos 48°$

Find the given values.

Use the symmetry of the graph to find the acute angle with the same numeric value.

Write the sign and acute angle of the trigonometric ratio.

The trigonometric graphs and equations can be transformed in the same way as quadratic and polynomial graphs.

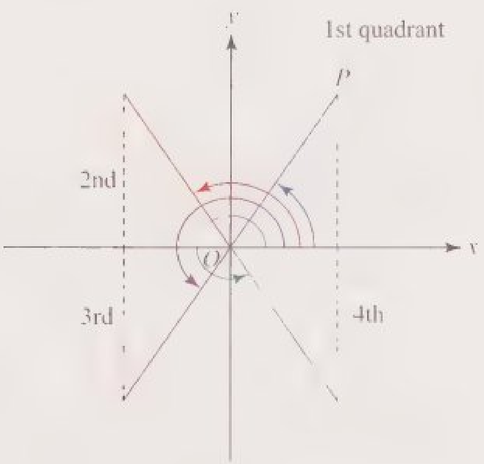
See Ch2.4 For a reminder of transformations of graphs.

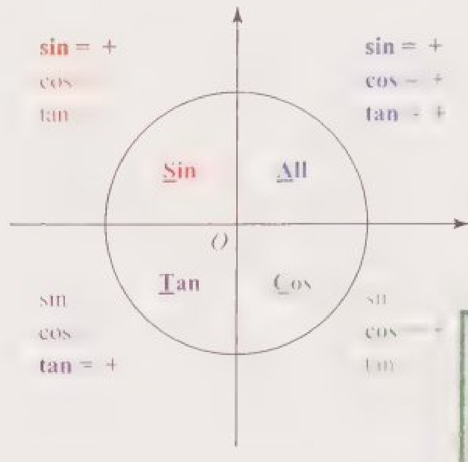
Another method for finding equivalent acute angles is to use a **quadrant diagram**.

Imagine a radius OP rotating about O through an angle θ from the positive x-axis. Whichever quadrant OP lies in, you can form an acute triangle with the x-axis as its base. Depending on the quadrant, the x and y coordinates of point P are positive or negative and so the sine, cosine and tangent of θ are also positive or negative.

You can use the word '**CAST**' as a mnemonic to help you remember where each ratio is positive.

CAST starts from the 4th quadrant and moves anticlockwise, as all the ratios are positive in the 1st quadrant.





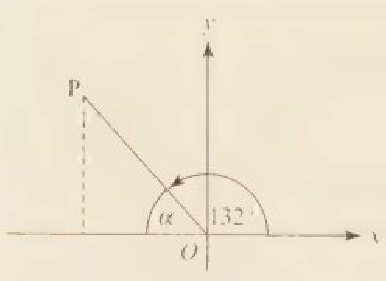
ICT Resource online

To experiment with graphical solutions of trigonometric equations, click this link in the digital book.

Example 4

Express **a** cos 132° **b** tan 683° in terms of acute angles.

a

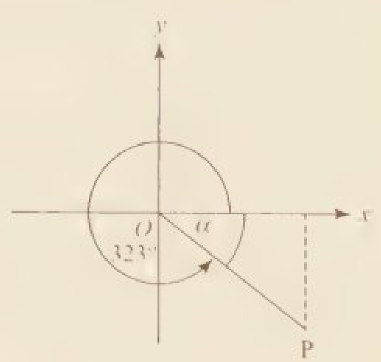


$\alpha = 180 - 132 = 48°$

132° is in the 2nd quadrant so cos 132° is negative.

$\cos 132° = -\cos 48°$

b $683° = 360° + 323°$



Draw a diagram showing the radius, the given angle, the triangle with the x-axis and the acute angle α

$\alpha = 360° - 323° = 37°$

Calculate the value of α

683° is in the 4th quadrant so tan 683° is negative.

Find the sign using CAST.

$\tan 683° = -\tan 37°$

Exercise 3.1A Fluency and skills

1 Use $\sin^2\theta+\cos^2\theta=1$ and $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ to calculate the value of $\sin\theta$ and $\tan\theta$, given that θ is acute and

a $\cos\theta=\dfrac{3}{5}$ b $\cos\theta=0.8$

c $\cos\theta=\dfrac{12}{13}$

2 Use the quadrant or graphical method to find these values in terms of acute angles.

a $\cos 190°$ b $\tan 160°$

c $\sin 340°$ d $\cos 158°$

e $\tan 215°$ f $\sin 285°$

Check your answers using the method that you didn't use the first time.

3 Copy and complete this table.

θ	$-90°$	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin\theta$						
$\cos\theta$						
$\tan\theta$						

4 a Describe the line and rotational symmetries of the graphs of

i $y=\cos\theta$ ii $y=\tan\theta$

b Sketch these graphs for $-360°\le x\le 360°$

You can check your sketches on a calculator.

i $y=\sin 3x$ ii $y=\cos x-1$

iii $y=\tan\dfrac{1}{2}x$

5 Simplify

a $\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$ b $\sqrt{\dfrac{1-\cos^2\theta}{1-\sin^2\theta}}$

c $\dfrac{\sqrt{1-\sin^2\theta}}{\cos\theta}$ d $\tan\theta\cos\theta$

e $\dfrac{\sin\theta}{\tan\theta}$ f $\sin\theta\cos\theta\tan\theta$

6 Express, in terms of acute angles,

a $\sin 380°$ b $\tan 390°$

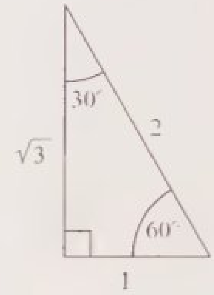
c $\cos 700°$ d $\tan(-42°)$

e $\cos(-158°)$ f $\sin(-203°)$

7 Solve these equations for $-180°\le\theta\le 180°$

a $\sin\theta=\cos\theta$ b $\sin\theta+\cos\theta=0$

8 Use the triangle to write these trigonometric ratios in surd form.



a $\sin 150°$ b $\cos 300°$

c $\tan 120°$ d $\sin 240°$

e $\cos(-60°)$ f $\tan(-150°)$

9 Use a calculator and give all the values of θ in the range $-360°$ to $360°$ for which

a $\sin\theta=0.4$ b $\tan\theta=1.5$

c $\cos\theta=-0.5$

10 Use a calculator to find the smallest positive angle for which

a $\sin\theta$ and $\cos\theta$ are both positive and $\sin\theta=0.8$

b $\sin\theta$ and $\tan\theta$ are both negative and $\sin\theta=-0.6$

11 Solve these equations for $0°\le\theta\le 360°$

Show your working.

a $4\sin\theta=3$ b $3\tan\theta=4$

c $2\sin\theta+1=0$ d $3\cos\theta+2=0$

e $\tan\theta+3=0$ f $7+10\sin\theta=0$

g $4\cos\theta=-3$ h $4+9\tan\theta=0$

MyMaths 2047, 2048, 2051, 2052, 2053, 2257 SEARCH

Reasoning and problem-solving

Strategy

To solve problems involving trigonometric ratios

1. Use trigonometric identities to simplify expressions.
2. Draw either a quadrant diagram or a trigonometric graph to show the information.
3. Use your knowledge of graphs, quadrant diagrams, symmetry and transformations to help you answer the question.

Example 5

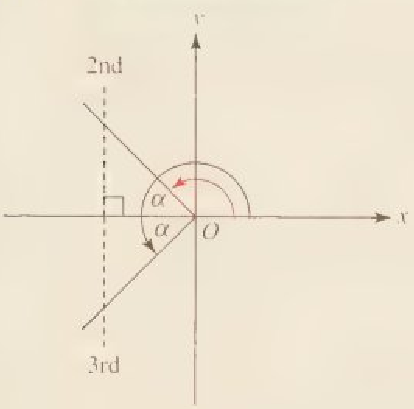
Solve $5\cos 2\theta + 3 = 0$ for $0° \le \theta \le 180°$. Show your working.

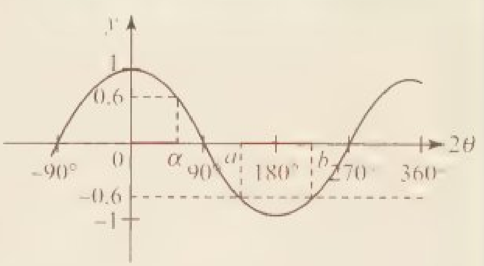
$5\cos 2\theta = -3$

$\cos 2\theta = -\frac{3}{5} = -0.6$ — (1) Rearrange and simplify.

For $0° \le 2\theta \le 360°$

either — or — (2) Draw either a quadrant diagram or a trigonometric graph.





$\cos 2\theta$ is negative in 2nd and 3rd quadrants

$\cos\alpha = 0.6$

$\alpha = 53.1°$

$2\theta = 180° - \alpha$ or $180° + \alpha$

$= 126.9°$ or $233.1°$

$\cos 2\theta = -0.6$ at a and b

$2\theta = \cos^{-1}(-0.6)$

$= 126.9° \ (= a)$ — Use a calculator to give the principal value.

Or $2\theta = 360° - 126.9°$

$= 233.1° \ (= b)$

$\theta = \frac{126.9°}{2}$

Or $\theta = \frac{233.1°}{2}$

$\theta = 63.5°$ or $116.6°$ — (3) Use the quadrant diagram or symmetry of graph to work out the values of θ

You can check your solution by solving the equation on a calculator.

Exercise 3.1B Reasoning and problem-solving

1 Solve these equations for $0° \le \theta \le 360°$

Show your working.

a $\sin(\theta + 30°) = \frac{1}{2}$ b $\cos(\theta - 30°) = \frac{1}{2}$

c $\tan(\theta + 20°) = 1$ d $2\sin(\theta + 30°) = -1$

2 Solve these equations for $0° \le \theta \le 180°$

a $2\sin 2\theta = 1$ b $3\tan 2\theta = 2$

c $5\cos 3\theta = 2$ d $5\sin 3\theta + 3 = 0$

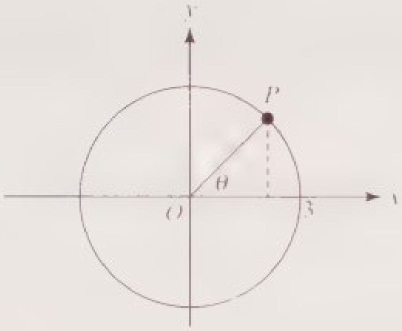
e $\frac{1}{2}\tan 2\theta + 3 = 0$ f $4\sin\left(\frac{1}{2}\theta\right) = 3$

3 Give full descriptions of any two transformations, which map the graph of

a $y = \sin\theta$ onto $y = \cos\theta$

b $y = \tan\theta$ onto itself.

4



a Given a circle, centre O and radius 3, write the coordinates of point P

b Show these coordinates satisfy the equation of the circle $x^2 + y^2 = r^2$ and write the value of r

c Another circle has the equation $4x^2 + 4y^2 = 25$. What is its radius?

5 A circle, with centre at the origin, passes through the point $(6, 8)$. What is its equation in its simplest form?

6 Solve these equations for $-180° \le \theta \le 180°$

Show your working.

a $2\sin\theta = \cos\theta$

b $4\cos\theta = 5\sin\theta$

c $3\sin 2\theta - \cos 2\theta = 0$

d $3\sin 2\theta = 2\tan 2\theta$

e $3\sin\theta + \tan\theta = 0$

f $\sin\theta\cos\theta - \cos\theta = 0$

7 Solve these equations for $0° \le \theta \le 360°$

a $\sin^2\theta - 2\sin\theta + 1 = 0$

b $\tan^2\theta - \tan\theta - 2 = 0$

c $2\sin\theta + 2 = 3\cos^2\theta$

d $2\cos^2\theta + \sin^2\theta = 2$

e $2\cos\theta + 2 = 4\sin^2\theta$

f $5\sin\theta - 4\cos^2\theta = 2$

8 The graph of $y = a\sin b\theta$ has a maximum value of 5 and a period of 45°. Find the values of a and b. Show your working.

9 The depth of water, h metres, at point P on the seabed changes with the tide and is given by $h = 3 + 2\sin(30° \times t)$, where t is the time in hours after midnight.

a What is the greatest and least depth of water at P?

b What is the period of the oscillation of the tide?

c At what times do the high tides occur on this day?

10 Solve these equations for $0 \le \theta \le 360°$

a $7\cos\theta + 6\sin^2\theta - 8 = 0$

b $4\cos^2\theta + 5\sin\theta = 3$

11 a Draw an accurate graph of the function $y = 2\cos\theta + 3\sin\theta$ for $-180° \le \theta \le 180°$

b Solve the equation $2\cos\theta + 3\sin\theta = 0$ for this range of θ by using

i your graph,

ii an algebraic method.

12 Prove these identities.

a $\cos^4 x - \sin^4 x \equiv \cos^2 x - \sin^2 x$

b $\tan x + \frac{1}{\tan x} \equiv \frac{1}{\sin x \cos x}$

Challenge

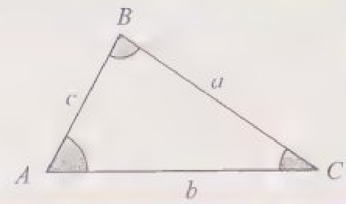
13 Prove that $\frac{1 - \tan^2 x}{1 + \tan^2 x} \equiv 1 - 2\sin^2 x$ and

hence solve the equation $\frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{1}{2}$

for $0 \le x \le 360°$

3.2 The sine and cosine rules

Fluency and skills



You can use the sine and cosine rules to calculate lengths and angles in any triangle — not just right-angled triangles.

When you know a pair of opposite sides and angles, you can calculate other sides and angles using the **sine rule**.

Key point

The sine rule states that, for triangle *ABC*,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

Use the left-hand version for sides and the right-hand one for angles.

Triangle *ABC* can be written ΔABC

Example 1

In ΔABC, angle $A=50°$, side $a=8$ cm and side $c=10$ cm. Calculate angles *B* and *C*, given that the triangle is acute.

$\frac{\sin C}{c}=\frac{\sin A}{a}$ — As *c* is known, use the sine rule to calculate angle *C* first.

$\frac{\sin C}{10}=\frac{\sin 50°}{8}$ — Substitute in the correct values.

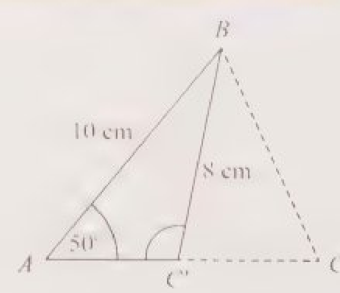
$\sin C=\frac{\sin 50°}{8}\times 10=0.9575...$ — Rearrange to solve for sin *C*. Do not round answers during a calculation.

Sine is positive between 0° and 180°, so there are two possible values of *C*

The given triangle is acute, so $C=73.2°$

Angle $B=180°-50°-73.2°=56.8°$ — Use the angle sum of a triangle.

The data in Example 1 ($A=50°$, $a=8$ cm, $c=10$ cm) can also describe an obtuse triangle where $C'=180°-73.2°=106.8°$ and $B=180°-50°-106.8°=23.2°$. This is an example of the **ambiguous case**, where the initial data gives two possible triangles.

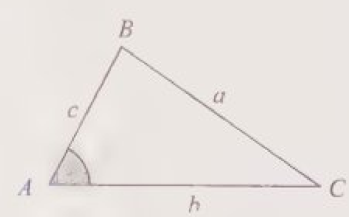


When you know two sides and the angle between them, you can use the **cosine rule** to calculate the third side. You can also use this rule to calculate angles when you know all three sides but no angles.

Key point

The cosine rule states that, for triangle *ABC*, $a^2=b^2+c^2-2bc\cos A$

Alternatively, $b^2=a^2+c^2-2ac\cos B$ or $c^2=a^2+b^2-2ab\cos C$



Example 2

In ΔABC, $a = 4$ cm, $b = 9$ cm and $c = 6$ cm. Calculate angle B

$b^2 = c^2 + a^2 - 2ca\cos B$ — You need to find angle B, so use the formula which has b^2 as the subject.

$9^2 = 6^2 + 4^2 - 2 \times 6 \times 4 \times \cos B$

$48\cos B = 36 + 16 - 81 = -29$

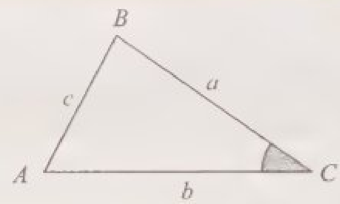
$\cos B = -\dfrac{29}{48}$ — Rearrange to make $\cos B$ the subject.

Angle $B = 127.2°$ — $\cos B$ is negative, so angle B is obtuse.

You can calculate the area of any triangle when you know any two sides and the angle between them.

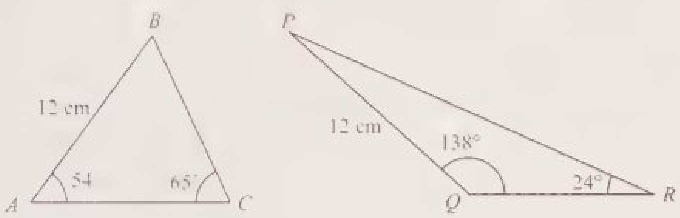
Key point

Area of triangle $ABC = \frac{1}{2}ab\sin C$



Exercise 3.2A Fluency and skills

1 Calculate the lengths BC and PR in these triangles.



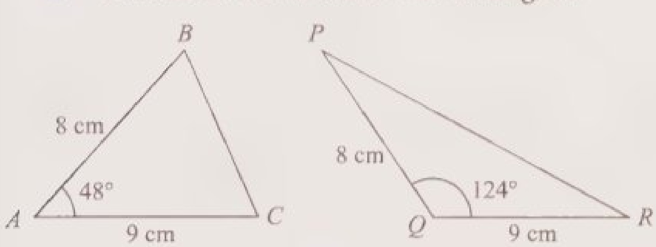
2 a Triangle ABC is acute with $AB = 9$ cm, $BC = 8$ cm and angle $A = 52°$. Calculate angle C

b Triangle EFG is obtuse with $EG = 11$ cm, $EF = 7$ cm and angle $G = 35°$. Calculate obtuse angle F

c Triangle HIJ is obtuse with $HI = 10$ cm, $IJ = 5$ cm and angle $H = 28°$. Calculate obtuse angle J

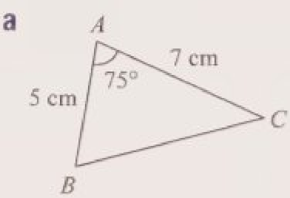
3 a Calculate the lengths BC and PR in these triangles.

b Calculate the areas of the triangles.

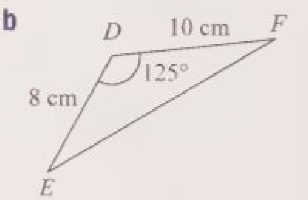


4 Calculate all the unknown sides and angles in these triangles. Give both solutions if the data is ambiguous.

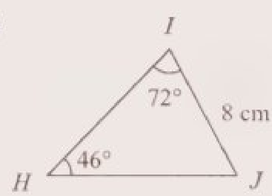
a



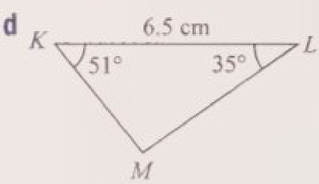
b



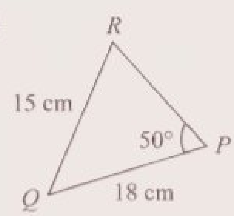
c



d



e



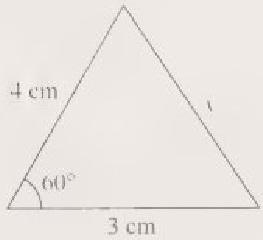
5 Calculate the unknown sides and angles in

a ΔABC where $a = 11.1$ cm, $b = 17.3$ cm and $c = 21.2$ cm

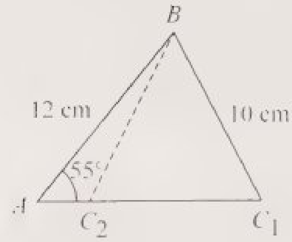
b ΔDEF where $d = 75.3$ cm, $e = 56.2$ cm and angle $F = 51°$

c ΔHIJ where with $h = 44.2$ cm, $i = 69.7$ cm and angle $J = 33°$

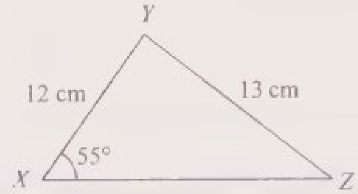
6 Calculate the length x and the area of the triangle, giving your answers as surds.



7 a In ΔABC, use the sine rule to show that there are two possible positions for vertex C (C_1 and C_2). Calculate the two possible sizes of angle C



b In ΔXYZ, calculate the size of angle Z and explain why it is the only possible value.



Reasoning and problem-solving

Strategy

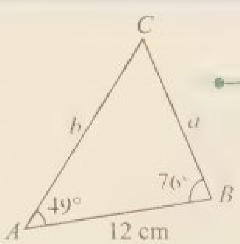
To solve problems involving sine and cosine rules or the area formula

1. Draw a large diagram to show the information you have and what you need to work out.
2. Decide which rule or combination of rules you need to use.
3. Calculate missing values and add them to your diagram as you solve the problem.

Example 3

In ΔABC, angle $A = 49°$, angle $B = 76°$ and $c = 12$ cm. Calculate the unknown sides and angles, and calculate the area of the triangle.

Angle $C = 180° - 76° - 49° = 55°$



Draw a diagram to show the information. (1)

The sine rule gives $\frac{a}{\sin 49°} = \frac{12}{\sin 55°}$

Choose the sine rule because side c and angle C are now both known. (2)

$a = \frac{12 \sin 49°}{\sin 55°} = 11.055...$

$= 11.1$ cm (to 3 sf)

Rearrange and calculate a (3)

The sine rule gives

$\frac{b}{\sin 76°} = \frac{12}{\sin 55°}$

$b = \frac{12 \times \sin 76°}{\sin 55°}$

$= 14.2$ cm (to 3 sf)

The cosine rule gives

$b^2 = 12^2 + 11.055^2 - 2 \times 12 \times 11.055 \times \cos 76°$

$= 202.03$

$b = 14.213...$

$= 14.2$ cm (to 3 sf)

You can use either the sine rule or the cosine rule to calculate b (2)(3)

You can decide which rule is easier to use.

The two unknown sides are 11.1 cm and 14.2 cm.

The area of triangle $= \frac{1}{2} bc \sin A$

$= \frac{1}{2} \times 14.2 \times 12 \times \sin 49° = 64.3$ cm^2

You could also use $\frac{1}{2} ac \sin B$ or $\frac{1}{2} ab \sin C$

Exercise 3.2B Reasoning and problem-solving

1 Calculate the area of ΔEFG, given that $e = 5$ cm, $f = 6$ cm and $g = 10$ cm.

2 a ΔABC has $AB = 5$ cm, $BC = 6$ cm and $AC = 7$ cm. Calculate the size of the smallest angle in the triangle.

b ΔEGF has $EF = 5$ cm, $FG = 7$ cm and $EG = 10$ cm. Calculate the size of the largest angle in the triangle.

3 ΔABC has $b = 4\sqrt{3}$ cm, $c = 12$ cm and angle $A = 30°$. Prove the triangle is isosceles.

4 A parallelogram has diagonals 10 cm and 16 cm long and an angle of 42° between them. Calculate the lengths of its sides.

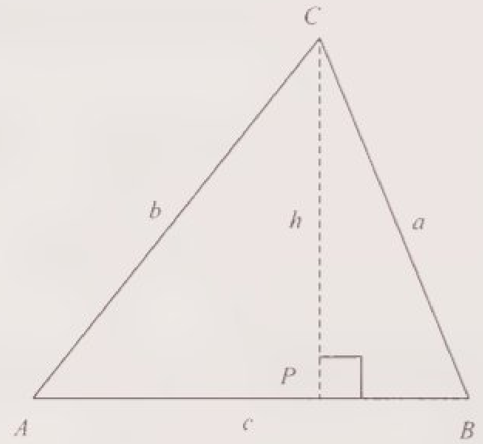
5 a ΔDEF has sides $d = 3x$, $e = x + 2$ and $f = 2x + 1$. If angle $D = 60°$, show that the triangle is equilateral. Calculate its area as a surd.

b ΔPQR has an area of $\frac{3}{4}$ m². If $p = x + 1$, $q = 2x + 1$ and angle $R = 30°$, what is the value of x?

6 a Find two different expressions for the height h using ΔACP and ΔBCP

Hence, prove the sine rule. Also prove that the area of $\Delta ABC = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B$

b Find expressions for CP, AP and BP in terms of the sides a, b, c and angle A. Hence, use Pythagoras' theorem to prove the cosine rule.

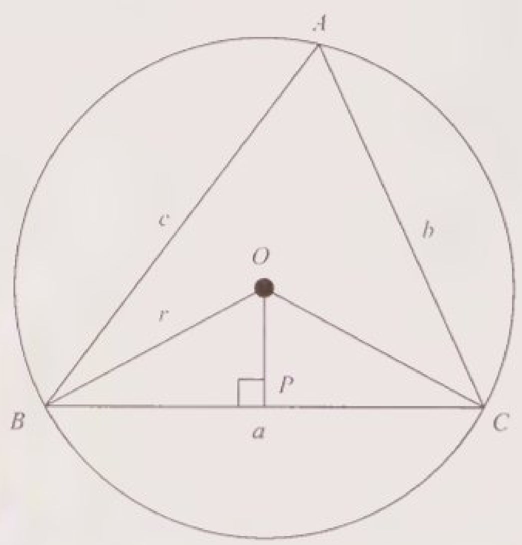


7 In ΔXYZ, $y = 2$ cm, $z = 2\sqrt{3}$ cm and angle $Y = 30°$. Prove that there are two triangles that satisfy this data and prove that one is isosceles and the other is right-angled.

8 The side opposite the smallest angle in a triangle is 8 cm long. If the angles are in the ratio 5 : 10 : 21, find the length of the other two sides.

9 Two circles, radii 7 cm and 9 cm, intersect with centres 11 cm apart. What is the length of their common chord?

10 The circumcircle of ΔABC has centre O and radius r, as shown in the diagram. Point P is the foot of the perpendicular from O to BC. Consider ΔBOP and prove that $2r = \frac{a}{\sin A}$. Hence, prove the sine rule.



11 A triangle has base angles of 22.5° and 112.5°. Prove that the height of the triangle is half the length of the base.

Challenge

12 In ΔXYZ, $x = n^2 - 1$, $y = n^2 - n + 1$ and $z = n^2 - 2n$
Prove that angle $Y = 60°$

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3 Summary and review

Chapter summary

- Sine, cosine and tangent are periodic functions. Their graphs have line and rotational symmetry.
- The sine, cosine and tangent of any angle can be expressed in terms of an acute angle.
- The sign and size of the sine, cosine or tangent of any angle can also be found using a sketch graph of the function.
- The two identities $\sin^2\theta + \cos^2\theta \equiv 1$ and $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ help you manipulate trigonometric expressions.
- The quadrant diagram can be used to find the sign and size of the sine, cosine or tangent of any angle. The mnemonic **CAST** helps you to remember in which quadrants the trigonometric ratios are positive.
- To solve a trigonometric equation, use identities to simplify it and then use a quadrant diagram or graph to find all possible angles.
- The sine and cosine rules are used to calculate unknown sides and angles in any triangle.
- The sine rule, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$, is used when you know a pair of opposite sides and angles.
- The cosine rule, $a^2 = b^2 + c^2 - 2bc\cos A$, is used when you know either two sides and the angle between them or all three sides.
- The area formula for a triangle, Area $= \dfrac{1}{2}ab\sin C$, uses two sides and the included angle.

Check and review

You should now be able to...	Try Questions
✔ Calculate the values of sine, cosine and tangent for angles of any size.	1, 3, 7
✔ Use the two identities $\sin^2\theta + \cos^2\theta \equiv 1$ and $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$, and recognise $x^2 + y^2 = r^2$ as the equation of a circle.	2, 6
✔ Sketch graphs of trigonometric functions and describe their main features.	4–5
✔ Solve various types of trigonometric equations.	8–12
✔ Use the sine and cosine rules and the area formula for a triangle.	13

1 Given that θ is acute, calculate the value of

a $\sin\theta$ and $\tan\theta$ when $\cos\theta = 0.8$ b $\cos\theta$ and $\tan\theta$ when $\sin\theta = \dfrac{5}{13}$

2 Simplify a $\dfrac{\cos\theta\sqrt{1-\cos^2\theta}}{1-\sin^2\theta}$ b $\dfrac{\tan\theta(1-\sin^2\theta)}{\cos\theta}$

3 Express, in terms of acute angles,

a $\sin 190°$ b $\tan 260°$ c $\cos 140°$ d $\tan 318°$ e $\sin 371°$

f $\cos 480°$ g $\tan(-150°)$ h $\cos(-200°)$ i $\sin(-280°)$

4 Find the maximum value of y and the period for these functions, showing your working.

a $y = 4\sin x$ b $y = 5\sin 2x$ c $y = 6\cos 5x$

5 Sketch the graphs of $y = \sin x$ and $y = \cos x$ for $0° \le x \le 180°$ on the same axes.

a Use your graph to solve the equation $\sin x = \cos x$

b Solve the same equation algebraically to check your solutions.

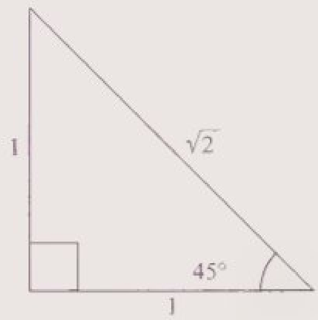
6 a Show that the point $P(2\cos\theta, 2\sin\theta)$ lies on a circle and find its radius.

b Show that the point $Q(1, \sqrt{3})$ lies on the circle and write the value of θ at Q

7 Use the triangle to write these trig ratios in surd form.

a $\sin 135°$ b $\cos 225°$ c $\tan 315°$

d $\cos 405°$ e $\sin(-135°)$ f $\tan(-225°)$



8 Give all the values of θ in the range $-360°$ to $360°$ for which

a $\cos\theta = 0.7$ b $\tan\theta = 2.5$ c $\sin\theta = -0.5$

9 Solve these equations for $0° \le \theta \le 360°$, showing your working.

a $3\sin\theta = 2$ b $2\tan\theta = 7$ c $2\cos\theta + 1 = 0$ d $\cos\theta\tan\theta = -0.5$ e $\sin\theta\tan\theta = \frac{1}{4}$

10 Solve these equations for $-180° \le \theta \le 180°$. Show your working.

a $4\sin\theta = 3\cos\theta$ b $4\sin\theta = 3\tan\theta$ c $3\sin^2\theta = \tan\theta\cos\theta$

d $\sin(\theta - 20°) = \frac{\sqrt{3}}{2}$ e $\cos(\theta + 30°) = \frac{1}{2}$ f $\tan(\theta - 10°) = -1$

11 Solve these equations for $0° \le \theta \le 180°$, showing your working.

a $3\sin 2\theta = 1$ b $5\tan 2\theta - 2 = 0$ c $5\sin 3\theta - 1 = 0$

d $3\cos 3\theta - 2 = 0$ e $3\sin 2\theta - \cos 2\theta = 0$ f $2\sin\left(\frac{1}{2}\theta\right) - \cos\left(\frac{1}{2}\theta\right) = 0$

12 Solve these equations for $0° \le \theta \le 360°$

a $2\cos^2\theta + \sin\theta = 1$ b $\cos^2\theta + \cos\theta = \sin^2\theta$ c $6\sin^2\theta + 5\cos\theta = 5$

d $\tan^2\theta = 2 + \frac{1}{\cos\theta}$ e $1 + \sin\theta\cos^2\theta = \sin\theta$ f $4\sin^2\theta = 2 + \cos\theta$

13 Calculate the side BC, the angle E, and the area of each triangle.

a ΔABC where $AC = 8$ cm, Angle $A = 42°$ and Angle $B = 56°$

b ΔDEF where $DF = 6$ cm, $EF = 11$ cm and Angle $D = 124°$

What next?

Score		
	0–6	Your knowledge of this topic is still developing. To improve, search in MyMaths for the codes: 2045–2048, 2051–2053, 2257
	7–10	You're gaining a secure knowledge of this topic. To improve, look at the InvisiPen videos for Fluency and skills (03A)
	11–13	You've mastered these skills. Well done, you're ready to progress! To develop your techniques, look at the InvisiPen videos for Reasoning and problem-solving (03B)

Click these links in the digital book

History

Trigonometry, as we know it today, was largely developed between the 16th and 18th centuries. However, the foundations of trigonometry were laid as long ago as the 3rd century BC.

Hipparchus (190 BC – 120 BC), a Greek mathematician and astronomer regarded by many as the founder of trigonometry, constructed the first known trigonometric tables based on the lengths of chords in circles. Contributions to the early development of the theory were made by scholars from a number of countries, including Greece, Turkey, India, Egypt and China.

ICT

Use a spreadsheet to calculate and compare the values of $\sin(x + y)$ and $\sin x \cos y + \cos x \sin y$ for different values of x and y

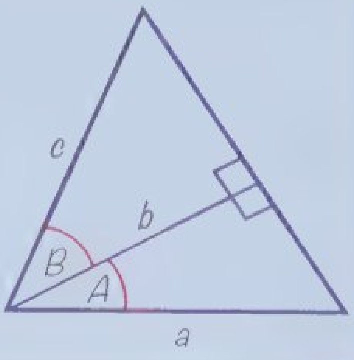
x	y	$\sin(x + y)$	$\sin x \cos y + \cos x \sin y$

Use a 3D graph plotter to draw the graphs of $\sin(x + y)$ and $\sin x \cos y + \cos x \sin y$
What do you notice about the two graphs?

Have a go

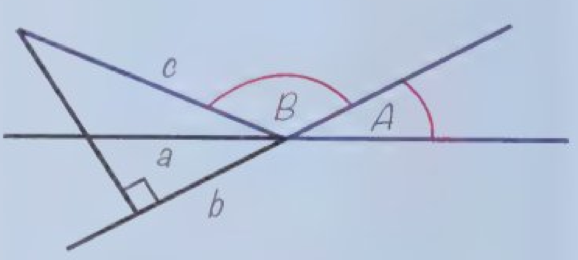
The result $\sin(A + B) = \sin A \cos B + \cos A \sin B$ has been known in various forms since ancient times.

Prove that this result is the case, where A and B are acute angles, by considering the areas of the three triangles shown in the diagram below.



Have a go

Adapt your proof from the other 'Have a go' for the case where one of the angles is acute and the other is obtuse.



1 a Solve $\cos x = 0.2$ for $0 \le x \le 360°$ Select the correct answer.

A $x = 78.5°$ B $x = 78.5°, 101.5°$ C $x = 78.5°, 281.5°$ D $x = 1.37°, 358.6°$ **[1 mark]**

b Solve $2\tan(x - 80°) = 3$ for $0 \le x \le 360°$. Select the correct answer.

A $x = 136.3, 316.3$ B $x = 160.5°, 340.5°$

C $x = 56.3°, 123.7°$ D $x = 56.3°, 236.3°$ [1]

2 For the curve with equation $y = \cos\frac{x}{2}$, select the coordinates of a

a maximum point,

A $(1, 0°)$ B $(0°, 1)$ and $(360°, 1)$ C $(1, 360°)$ D $(0°, 1)$ [1]

b minimum point.

A $(180°, -1)$ B $(-1, 180°)$ C $(360°, -1)$ D $(180°, 0)$ [1]

3 Sketch these graphs for $-180° \le x \le 180°$

a $y = \sin 2x$ [3]

b $y = \tan(x - 20°)$ [4]

On each diagram, show the coordinates where the curve crosses the x-axis and give the equations of any asymptotes.

4 $f(x) = \sin(x + 45°)$ for $0 \le x \le 360°$

a Sketch the graph of $y = f(x)$ and label the coordinates of intersection with the axes. [3]

b Write down the coordinates of the minimum and maximum points in this interval. [3]

c Solve the equation $\sin(x + 45°) = 0.3$ for x in the interval $0° \le x \le 360°$ [4]

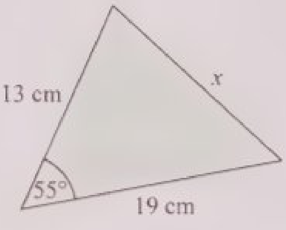
5 The triangle DEF has $DE = 8$ m, $EF = 6$ m and $DF = 7$ m.

a Calculate the size of $\angle DEF$ [3]

b Calculate the area of the triangle. [3]

6 A triangle has side lengths 19 cm, 13 cm and x and angle 55° as shown.

Calculate the size of x [3]



7 An equilateral triangle has area $\sqrt{3}$ square units.

Calculate the side lengths of the triangle. [3]

8 Solve the equations for θ in the interval $-180° \le \theta \le 180°$

a $2\sin(x - 10) = -0.4$ [4]

b $\tan 3x = 0.7$ [6]

9 The curve C has equation $y = \tan(x - \alpha)$ with $-180° \le x \le 180°$ and $0° < \alpha < 45°$

a Sketch C and label the points of intersection with the x-axis. [3]

b Write down the equations of the asymptotes. [2]

c Solve the equation $\tan(x - \alpha) = \sqrt{3}$ for $-180° \le x \le 180°$ [4]

10 $f(x) = k\cos x$ where k is a positive constant.

a Sketch the graph of $y = f(x)$ for $0 \leq x \leq 360°$
Label the points of intersection with the coordinate axes. [3]

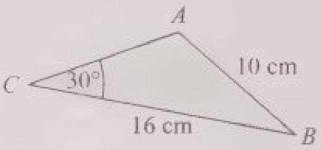
b Given that $k = 3$, solve the equation $k\cos x = \sin x$ for $0° \leq x \leq 360°$ [4]

11 $f(x) = \cos 2x$, $g(x) = 1 - \frac{x}{45}$

a Sketch $y = f(x)$ and $y = g(x)$ on the same axes for $0° \leq x \leq 360°$ [5]

b How many solutions are there to $f(x) = g(x)$? Justify your answer. [1]

12 A triangle ABC has $AB = 10$ cm, $BC = 16$ cm and $\angle BCA = 30°$



a Calculate the possible lengths of AC [5]

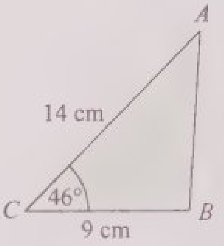
b What is the minimum possible area of the triangle? [3]

13 In the triangle CDE, $CD = 9$ cm, $CE = 14$ cm and $\angle CDE = 54°$

a Calculate the size of $\angle DEC$ [3]

b Explain why there is only one possible value of $\angle DEC$ [2]

14 In the triangle ABC, $BC = 9$ cm, $CA = 14$ cm and $\angle BCA = 46°$



a Calculate the length of side AB [3]

b Calculate the size of the largest angle in the triangle. [3]

15 A triangle has side lengths 12 cm, 8 cm and 6 cm.

Calculate the size of the largest angle in the triangle. [4]

16 Solve the inequality $\sin x > \frac{\sqrt{2}}{2}$ for $0 \leq x \leq 360°$ [4]

17 a Show that $\cos\theta + \tan\theta\sin\theta \equiv \frac{1}{\cos\theta}$ [3]

b Hence solve $\cos\theta + \tan\theta\sin\theta = 2.5$ for $-180° \leq \theta \leq 180°$ [3]

18 Solve, for $0° \leq x \leq 360°$, the equations

a $\sin^2 x = 0.65$ [5]

b $\tan^2 x - 2\tan x - 3 = 0$ [4]

19 Solve, for $0 \leq x \leq 180°$, the equation $3\cos^2 2x = 2\sin 2x + 3$ [8]

20 $f(\theta) = 2\tan 3\theta + 5\sin 3\theta$

Find all the solutions to $f(\theta) = 0$ in the range $0° \leq \theta \leq 180°$ [9]

21 Given that $\sin x = \frac{1}{5}$ and x is acute, find the exact value of

a $\cos x$ [2] b $\tan x$ [3]

Give your answers in the form $a\sqrt{b}$ where a is rational and b is the smallest possible integer.

22 $2\cos^2\theta - k\sin\theta = 2 - k$

a Find the range of values of k for which the equation has no solutions. [6]

b Find the solutions in the range $0° \leq \theta \leq 360°$ when $k = 1$ [4]

23 Find an expression for $\tan\theta$ in terms of α, given that θ is acute and $\sin\theta = \alpha$ [3]

24 Solve the equation $\sin^4 x - 5\cos^2 x + 1 = 0$ for $0° \leq x \leq 360°$ [8]

25 The area of an isosceles triangle is 400 cm^2

Calculate the perimeter of the triangle given that one of the angles is $150°$ [7]