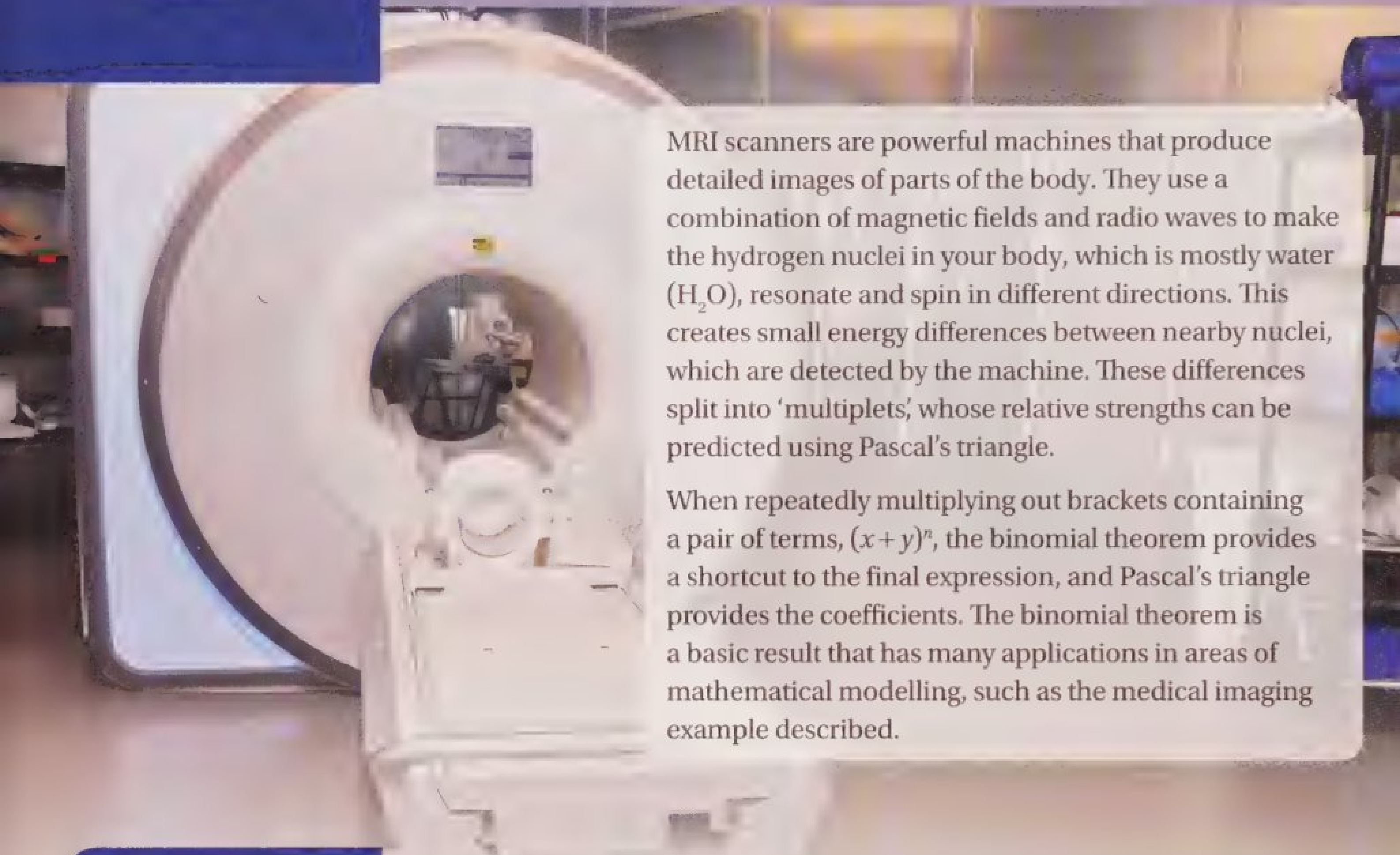
2 Polynomials and the binomial theorem



MRI scanners are powerful machines that produce detailed images of parts of the body. They use a combination of magnetic fields and radio waves to make the hydrogen nuclei in your body, which is mostly water (H_2O), resonate and spin in different directions. This creates small energy differences between nearby nuclei, which are detected by the machine. These differences split into 'multiplets,' whose relative strengths can be predicted using Pascal's triangle.

When repeatedly multiplying out brackets containing a pair of terms, $(x+y)^n$, the binomial theorem provides a shortcut to the final expression, and Pascal's triangle provides the coefficients. The binomial theorem is a basic result that has many applications in areas of mathematical modelling, such as the medical imaging example described.

Orientation

What you need to know	What you will learn	What this leads to
KS4 • To simplify and manipulate algebraic expressions, including collecting like terms and use of brackets.	• To manipulate, simplify and factorise polynomials. • To understand and use the binomial theorem. • To divide polynomials by algebraic expressions. • To understand and use the factor theorem. • To analyse a function and sketch its graph.	**Ch4 Differentiation and integration** Using calculus for curve sketching. **Ch10 Probability and discrete random variables** The binomial probability distribution. **Ch13 Sequences** Binomial expansions. Position-to-term and term-to-term rules.

MyMaths Practise before you start: 1150, 1155, 1156

2.1 Expanding and factorising

Fluency and skills

A **polynomial** is an algebraic expression that can have constants, variables, **coefficients** and **powers** (also known as **exponents**), all combined using addition, subtraction, multiplication and division.

> The highest power in a polynomial is called its **degree**. **Key point**

All quadratics are of degree two.

$$y^3 + 4xy^2 + 5x - \frac{1}{7}xy - 9$$

3 and 2 are powers (or exponents). This polynomial is of degree 3

(+)4, (+)5 and $-\frac{1}{7}$ are coefficients.

9 is a constant.

x and y are variables.

You can simplify polynomials by collecting (adding or subtracting) **like terms**. You must *never* attempt to simplify a polynomial by dividing by a variable and the exponent of a variable can only be 0, 1, 2, 3, ...etc.

You can manipulate polynomials by **expanding**, **simplifying** and **factorising** them.

Example 1

Expand and simplify $(3x+2y)^2 - (2x-3y)^2$

$(3x+2y)^2 = (3x+2y)(3x+2y)$ — Expand the brackets before adding or subtracting polynomials.

$= 9x^2 + 12xy + 4y^2$ — Multiply each term in the first bracket by each term in the second bracket.

$(2x-3y)^2 = 4x^2 - 12xy + 9y^2$ so $-(2x-3y)^2 = -4x^2 + 12xy - 9y^2$

$9x^2 + 12xy + 4y^2$

$-4x^2 + 12xy - 9y^2$ — You may find it useful to write like terms vertically under each other.

$= 5x^2 + 24xy - 5y^2$ — Collect like terms to simplify the polynomial.

> A statement that is true for all values of the variable(s) is called an **identity**. **Key point**
>
> You write an identity using the symbol ≡

For example, $15x^3 + 8x^2 - 26x + 8 \equiv (3x^2+4x-2)(5x-4)$ is true for all values of x

It follows that $(3x^2+4x-2)$ and $(5x-4)$ are **factors** of $15x^3 + 8x^2 - 26x + 8$

Factorising is the opposite process to expanding brackets.

You can factorise polynomials by comparing coefficients.

Example 2

$(4x-5)$ is a factor of the polynomial $12x^3+21x^2-61x+20$

Factorise the polynomial completely.

$12x^3+21x^2-61x+20 \equiv (4x-5)(Ax^2+Bx+C)$ — Use the fact that $4x-5$ is a factor to write an identity.

$(4x-5)(Ax^2+Bx+C)$ — To expand, multiply each term in the first bracket by each term in the second bracket.

$\equiv 4Ax^3+4Bx^2+4Cx$

$\quad -5Ax^2-5Bx-5C$ — To collect like terms write them under each other.

$\equiv 4Ax^3+(4B-5A)x^2+(4C-5B)x-5C$

This is identical to $12x^3+21x^2-61x+20$

so the coefficients must all be the same.

so $4A=12$ ① — Equate and compare coefficients for x^3, x^2, x and compare the constants.

$4B-5A=21$ ②

$4C-5B=-61$ ③

$-5C=20$ ④

$A=3$ and $C=-4$ — Rearrange ① and ④

$4B-5\times3=21$ — Substitute $A=3$ into ②

$4B=36$ so $B=9$

$4(-4)-5(9)=-61$ ✓ — Check by substituting the values into ③

So $12x^3+21x^2-61x+20 \equiv (4x-5)(3x^2+9x-4)$ — State your answer clearly and check it by expanding the brackets. $(3x^2+9x-4)$ cannot be factorised so this is the final answer.

Exercise 2.1A Fluency and skills

1 Write the degree of each of these expressions.

a $3-2x+x^2$ b $1-3x+5x^4$ c $2x^2-x+1-4x^3$

2 Expand and simplify each of these expressions.

a $2x(3x+8)$ b $2x(3x^2+8x-9)$ c $(3y+2)(4y-7)$

d $3y(4y^2+8y-7)$ e $(t-5)^2$ f $(t+3)(t-5)^2$

3 Expand and simplify each of these expressions.

a $(x+4)^2+(x-4)^2$ b $(5p+q)^2-(5p-q)^2$

4 Factorise each of these expressions.

a $4m^3+6m^2$ b $16n^4-12n$ c $5p^4-2p^2+6p$ d $9y^2-15xy$

e $6x^2-3xy+9x$ f $7yz-21z^3$ g $4e(e-2f)-12ef$ h p^2-100

i $6q(3-2q)+9q$ j $\frac{y}{5}-\frac{y^2}{15}+\frac{3y}{25}$ k $(d+1)(d+3)+(d+1)(d-5)$

l $w(2w+3)(3w+9)+w(2w-11)(2w+3)$

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5 Fully factorise these expressions.

a $4m^3+4m^2-15m$ b $7n^3-15n^2+2n$

6 Factorise this expression $3x(x+2)^2+(x+2)(5x^2+2x-6)$

7 Expand and simplify these expressions.

a $(5p+4q)^2-(5p-4q)^2$ b $(x+y+z)^2-(x-y-z)^2$

c $(x\sqrt{3}+4)^2+(x\sqrt{3}-4)^2$ d $(x\sqrt{5}+4)^2+(x\sqrt{3}-4)^2$

8 a (x^2+3x+9) is a factor of x^3+2x^2+6x-9. Work out the other factor.

b (x^2-2x+3) is a factor of $2x^3-11x^2+20x-21$. Work out the other factor.

c $(y^2+2y-15)$ is a factor of $2y^3+3y^2-32y+15$. Work out the other factor.

d $(z-2)$ is a factor of z^3+z^2-2z-8. Work out the other factor.

e $(2a+5)$ is a factor of $6a^3+7a^2-2a+45$. Work out the other factor.

f (x^2-4x+7) is a factor of $2x^3-5x^2+2x+21$. Factorise the polynomial fully.

g (k^2-3k+1) is a factor of $k^4+3k^3-24k^2+27k-7$. Work out the other factors.

Reasoning and problem-solving

Strategy

To factorise polynomials

1. Look for obvious common factors and factorise them out.
2. Write an identity and expand to compare coefficients.
3. Write your solution clearly and use suitable units where appropriate.

Example 3

The volume of a cylinder is $y^2-25y+24$ ft^3

The base area is $(y-1)$ ft^2

Write an expression for its height.

Let the height be $(Ay+B)$ ft

So $y^2-25y+24 \equiv (Ay+B)(y-1)$

$y^2-25y+24 \equiv Ay^2+By-Ay-B$

$y^2-25y+24 \equiv Ay^2+(B-A)y-B$

So $A=1$ ①

$(B-A)=-25$ ②

$-B=24 \therefore B=-24$ ③

$-24-1=-25$ ✓

So the height is $(y-24)$ ft

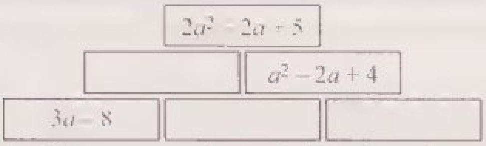
The height must be linear because it multiplies with $(y-1)$ to give a quadratic.

(2) Write an identity and expand to compare coefficients.

Check by substituting into ②

(3) Write your solution clearly and use suitable units.

Exercise 2.1B Reasoning and problem-solving

1 

In this pyramid, each block is the sum of the two blocks vertically beneath.

Copy and complete the pyramid.

2 A square has side length $(4b - 7a)$ cm.

Write an expression for its area in expanded form.

3 A cuboid has sides of length $(c + 2)$, $(2c - 1)$ and $(3c - 7)$ cm. Write an expression for its volume in expanded form.

4 A square hole of side length $(a + 2)$ cm is cut from a larger square of side length $(2a + 5)$ cm. Without expanding any brackets, write the remaining part of the large square as a pair of factors.

5 A rectangle, sides $2a$ cm by a cm, has a square of side x cm cut from each corner. The sides are then folded up to make an open box. Work out the volume of this box.

6 A ball is thrown from ground level and its height, h ft, at time t s is given by the polynomial $h = 25t - 5t^2$

a When does the ball next return to ground level?

b What is the maximum height reached by the ball?

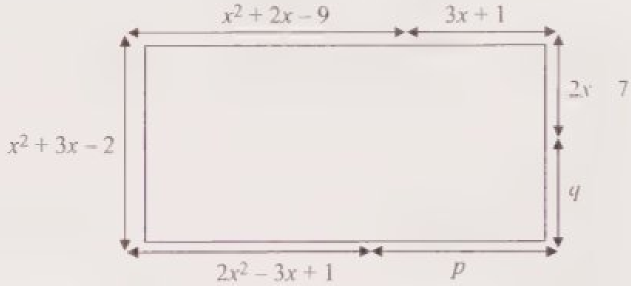
7 A body moves along a straight line from a point O where its position, x metres at time, t seconds is given by the equation $x = 3t^3 - 28t^2 + 32t$. Its velocity, v m s^{-1} and acceleration, a m s^{-2} at time t are given by the equations $v = 9t^2 - 56t + 32$ and $a = 18t - 56$

a Find the values of t when the body is at O, and find its velocity and acceleration at these times.

b Find the distance of the body from O and its velocity when its acceleration is zero.

c Find the value(s) of t when its velocity is zero, and find its acceleration at these times.

8 A rectangle has the dimensions shown. All lengths are given in centimetres.



a Find p and q in terms of x

b Write expanded expressions for the rectangle's perimeter and area in terms of x

9 Kia has lost her calculator. Show how she can complete these calculations.

a $66.89^2 - 33.11^2$

b $(\sqrt{8})^3 - (\sqrt{2})^3$

10 A cuboid has volume $(2h^3 + 3h^2 - 23h - 12)$ cm³. Its length is $(h + 4)$ cm and its width is $(h - 3)$ cm Work out the height of the cuboid.

11 The area of a trapezium is given by the polynomial $(2s^3 - 17s^2 + 41s - 30)$ cm². The perpendicular height is $(4s - 6)$ cm. Write an expression for the sum of the parallel sides.

12 The area of an ellipse is given by the formula πab, where a and b are half the lengths of the axes of symmetry. The area is $\pi(6t^3 - 5t^2 + 15t + 14)$ and $a = (3t + 2)$. Write an expression for b

Challenge

13 $V = \pi l[r^2 - (r - a)^2]$ is the volume of a circular pipe.

Find an expression for V in terms of p when $l = 4p + 5$, $r = 3p - 4$ and $a = p + 1$

2.2 The binomial theorem

Fluency and skills

You can expand $(1 + x)^n$ where $n = 0, 1, 2, 3, \ldots$

EXPANSION	COEFFICIENTS
$(1+x)^0 \equiv 1$	1
$(1+x)^1 \equiv 1 + 1x$	1 1
$(1+x)^2 \equiv 1 + 2x + 1x^2$	1 2 1
$(1+x)^3 \equiv 1 + 3x + 3x^2 + 1x^3$	1 3 3 1
$(1+x)^4 \equiv 1 + 4x + 6x^2 + 4x^3 + 1x^4$	1 4 6 4 1
$(1+x)^5 \equiv 1 + 5x + 10x^2 + 10x^3 + 5x^4 + 1x^5$	1 5 10 10 5 1

The coefficients form a pattern known as **Pascal's triangle**.

Each coefficient in the triangle is the sum of the two coefficients above it.

Pascal's Triangle was published in 1654, but was known to the Chinese and the Persians in the 11th century.

Example 1

Use Pascal's triangle to write the expansion of $(1 + 2y)^6$ in ascending powers of y

The coefficients are 1, 6, 15, 20, 15, 6, 1 — Write down the 6th row of Pascal's triangle.

$(1 + (2y))^6$ — Use the expansion of $(1 + x)^n$, substituting $2y$ for x

$\equiv 1 + 6(2y) + 15(2y)^2 + 20(2y)^3 + 15(2y)^4 + 6(2y)^5 + (2y)^6$

$\equiv 1 + 12y + 60y^2 + 160y^3 + 240y^4 + 192y^5 + 64y^6$

Replacing 1 with a and x with b gives the **binomial expansion** $(a + b)^n$ where $n = 0, 1, 2, 3, \ldots$

A binomial expression has two terms.

As n increases you can see that again the coefficients form Pascal's triangle.

See Ch 10.2
For more uses of the binomial expansion.

$$(a+b)^0 \equiv 1$$
$$(a+b)^1 \equiv 1a + 1b$$
$$(a+b)^2 \equiv 1a^2 + 2ab + 1b^2$$
$$(a+b)^3 \equiv 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
$$(a+b)^4 \equiv 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$
$$(a+b)^5 \equiv 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^5$$

In each expansion, the power of a starts at n and decreases by 1 each term, so the powers are n, $n-1, n-2, \dots, 0$

The power of b starts at 0 and increases by 1 each term, so the powers are $0, 1, 2, \dots, n$

The sum of the powers of any individual term is always n

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Example 2

Expand $(2+3t)^4$

$$(2+3t)^4$$

$$\equiv 2^4 + 4\times 2^3 \times (3t) + 6\times 2^2 \times (3t)^2 + 4\times 2\times (3t)^3 + (3t)^4$$

$$\equiv 16 + 96t + 216t^2 + 216t^3 + 81t^4$$

Use Pascal's triangle and the expansion of $(a+b)^4$ substituting 2 for a and $3t$ for b

It would be impractical to use Pascal's triangle every time you need to work out a coefficient—say, for example, you want to find the coefficient of x^6 in $(x+a)^{10}$

There is a general rule for finding this coefficient without needing to write out Pascal's triangle up to the tenth row.

Note that the first coefficient in each row is the 0th coefficient.

Key point

The rth coefficient in the nth row is ${}^nC_r \equiv \dfrac{n!}{(n-r)!r!}$

${}^nC_r$ is sometimes written as $\begin{pmatrix} n \\ r \end{pmatrix}$ or ${}_nC_r$

Key point

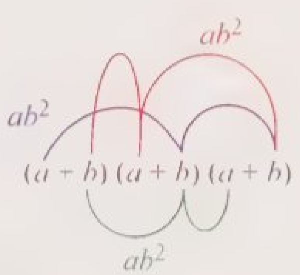
$n!$ stands for the product of all integers from 1 to n. You read it as n **factorial**.

For example, $6! = 6\times 5\times 4\times 3\times 2\times 1 = 720$

Look for the factorial button on your calculator. It may be denoted $x!$

${}^nC_r$ is the choose function and you read it as 'n choose r'. It gives the number of possible ways of choosing r elements from a set of n elements when the order of choosing does not matter. For example, the number of combinations in which you can choose 2 balls from a bag of 5 balls is ${}^5C_2$

You use the choose function because there are several ways of getting certain powers from an expansion. For example, there are 3 ways of getting ab^2 from the expansion of $(a+b)^3$: a from either the first, second or third bracket and b from the other two brackets in each case. The term in ab^2 for the expansion of $(a+b)^3$ is therefore ${}^3C_1\, ab^2 = 3ab^2$



Example 3

A term in the expansion of $(y+2x)^9$ is given by ky^3x^6

Find the value of k

$^9C_6 \times y^3 \times (2x)^6 \equiv 84 \times y^3 \times 64x^6$ — Use your calculator to find $^9C_6$ and work out 2^6

$\equiv 5376y^3x^6$

$k = 5376$ — Simplify to find the value of k

The formula for the binomial expansion of $(a+b)^n$ is sometimes called the **binomial theorem**.

Key point

$$(a+b)^n \equiv a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_ra^{n-r}b^r + \ldots + b^n$$

For the expansion of $(1+x)^n$ this gives

Key point

$$(1+x)^n \equiv 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

Example 4

Write the term in z^4 in the expression $(2z-1)^{15}$. Simplify your answer.

Take $a = 2z$ and $b = -1$

$^{15}C_{11}(2z)^{15-11}(-1)^{11}$ — The powers add to 15 so the second power must be 11. Use the coefficient $^{15}C_{11}$

$\equiv 1365 \times 16z^4 \times (-1)$ — $(2z)^4 =$ [illegible]

$= -21840z^4$

Exercise 2.2A Fluency and skills

1 Calculate the values of

a $5!$ b $7!$ c $11!$

2 Calculate the values of

a $^5C_2$ b $^9C_3$ c $^{11}C_7$ d $^{13}C_8$

3 Work out the values of

a $\binom{5}{3}$ b $\binom{10}{1}$ c $\binom{13}{5}$ d $\binom{20}{6}$

4 Use Pascal's triangle to find the expansions of each of these expressions.

a $(1+3x)^3$ b $\left(1-\frac{z}{2}\right)^5$

c $\left(1-\frac{m}{3}\right)^4$ d $\left(1+\frac{3x}{2}\right)^5$

5 Find the first four terms of these binomial expansions in ascending powers of x

a $(1+x)^8$ b $(1-3x)^7$

c $(1+2x)^9$ d $(2-3x)^6$

e $(x-2)^8$ f $(2x-1)^{10}$

6 Use Pascal's triangle to expand each of these expressions.

a $(2-4y)^3$ b $(3b+5)^4$ c $\left(4z-\frac{y}{3}\right)^5$

7 Find the first three terms of these binomial expansions in descending powers of x

a $(2+x)^6$ b $(1-2x)^8$

c $(3-x)^9$ d $(x+4)^7$

e $(2x+3)^{10}$ f $\left(\frac{x}{2}+4\right)^{11}$

8 Use the binomial theorem to expand each of these expressions.

a $(2+3t)^4$ **b** $(3-2p)^4$

c $(4p+3q)^5$ **d** $(3p-4q)^5$

e $(3z-2)^4$ **f** $\left(2z-\frac{1}{2}\right)^6$

g $\left(2+\frac{2x}{3}\right)^3$ **h** $\left(\frac{r}{3}+\frac{s}{4}\right)^6$

i $\left(\frac{x}{2}+\frac{y}{3}\right)^7$

9 Find the terms indicated in each of these expansions and simplify your answers.

a $(p+5)^5$ term in p^2

b $(4+y)^9$ term in y^5

c $(3+q)^{12}$ term in q^7

d $(4-3m)^5$ term in m^3

e $(2z-1)^{15}$ term in z^4

f $\left(z+\frac{3}{2}\right)^8$ term in z^6

g $(3x+4y)^5$ term in y

h $(2a-3b)^{10}$ terms in **i** a^5 and **ii** b^4

i $\left(4p+\frac{1}{4}\right)^3$ term in p^2

j $\left(4a-\frac{3b}{4}\right)^{11}$ terms in **i** a^5 and **ii** b^5

k $\left(\frac{a}{2}-\frac{2b}{3}\right)^{11}$ terms in **i** a^7 and **ii** b^5

10 Use the binomial theorem to expand each of these expressions.

a $(c^2+d^2)^4$ **b** $(v^2-w^2)^5$

c $(2s^2+5t^2)^3$ **d** $(2s^2-5t^2)^3$

e $\left(d+\frac{1}{d}\right)^3$ **f** $\left(2w+\frac{3}{w}\right)^4$

11 Use the binomial theorem to expand each of these brackets.

a $\left(x+\frac{2}{x}\right)^3$ **b** $(x^2-2)^4$

c $\left(x^2-\frac{1}{x}\right)^5$ **d** $\left(\frac{1}{x^2}+3x\right)^6$

12 Expand and simplify each of these expressions.

a $3x(2x-5)^5$ **b** $(2+x)^4(1+x)$

13 Expand and simplify each of these expressions.

a $(5-2x)^3+(3+2x)^4$

b $(1+3x)^5-(1-4x)^3$

14 Expand and fully simplify each of these expressions. Show your working.

a $(2+\sqrt{3})^4+(1-\sqrt{3})^4$

b $(1-\sqrt{5})^5-(2\sqrt{5}+3)^3$

15 Write down the first four terms of the expansion of each of these in ascending powers of x

a $(1+2x)^n$ **b** $(1-3x)^n$

where $n\in\mathbb{N}, n>3$

16 a Expand $(1+4x)^6$ in ascending powers of x up to and including the term in x^2

b Use your answer to part **a** to estimate the value of $(1.04)^6$

17 a Expand $(1-2x)^7$ in ascending powers of x up to and including the term in x^3

b Use your answer to part **a** to estimate the value of $(0.99)^7$

18 Use the binomial expansion to simplify each of these expressions. Give your final solutions in the form $a+b\sqrt{2}$

a $(1+\sqrt{2})^3$ **b** $(1-\sqrt{2})^5$

c $(3+2\sqrt{2})^4$ **d** $(\sqrt{2}-2)^6$

e $\left(1-\frac{1}{\sqrt{2}}\right)^3$ **f** $\left(\frac{\sqrt{2}}{3}+3\right)^4$

19 Use the binomial expansion to fully simplify each of these expressions.

Give your final answers in surd form.

a $(1+\sqrt{3})^4$ **b** $(1-\sqrt{5})^6$

c $(5-\sqrt{7})^5$ **d** $(2\sqrt{6}+5)^3$

e $(\sqrt{2}+\sqrt{6})^4$ **f** $(\sqrt{3}-\sqrt{2})^6$

Reasoning and problem-solving

Strategy

To construct a binomial expansion

1. Create an expression in the form $(1 + x)^n$ or $(a + b)^n$
2. Use Pascal's triangle or the binomial theorem to find the required terms of the binomial expansion.
3. Use your expansion to answer the question in context.

Example

A football squad consists of 13 players. Use the formula ${}^nC_r \equiv \dfrac{n!}{(n-r)!r!}$ to show that there are 78 possible combinations of choosing a team of 11 players from this squad.

$${}^{13}C_{11} = \frac{13!}{(13-11)!11!}$$

$$= \frac{13 \times 12 \times 11 \times 10 \times \ldots \times 2 \times 1}{2! \times 11 \times 10 \times \ldots \times 2 \times 1}$$

$$= \frac{13 \times 12}{2!}$$

(2) Cancel the common factor 11!

$$= \frac{156}{2} = 78$$

Example 6

a Using the first *three* terms of the binomial expansion, estimate the value of 1.003^8

b By calculating the fourth term in the expansion show that the estimate from part **a** is accurate to 3 decimal places.

a $1.003^8 = (1 + 0.003)^8$

(1) Rewrite in the form $(1 + x)^n$

First 3 terms

$$= 1 + nx + \frac{n(n-1)}{2!}x^2$$

(2) Use the first 3 terms of the general expansion.

$$= 1 + 8(0.003) + 28(0.003)^2$$

$$= 1 + 0.024 + 0.000252$$

$$= 1.024252 \; (= 1.024 \text{ to 3 sf})$$

(3) [illegible]

b $\dfrac{n(n-1)(n-2)}{3!}x^3 = 56(0.003)^3$

$$= 0.000001512$$

Adding this term will not affect the first three decimal places.

Exercise 2.2B Reasoning and problem-solving

1 How many possible ways are there to pick a 7's rugby team from a squad of 10 players?

2 How many possible ways are there to choose half of the people in a group of 20?

3 A cube has side length $(2s - 3w)$. Use the binomial expansion to find its volume.

4 Use Pascal's triangle to find the value of

a 1.05^6 correct to six decimal places,

b 1.96^3 correct to four decimal places.

5 Use the binomial theorem to work out the value of

a 1.015^5 correct to 4 decimal places,

b $\left(\frac{199}{100}\right)^{10}$ correct to five significant figures.

6 Use the binomial theorem to work out the value of $\left(\frac{13}{4}\right)^5$ correct to five decimal places.

7 Work out the exact value of the middle term in the expansion of $\left(\sqrt{3}+\sqrt{5}\right)^{10}$

8 a Find the coefficient of x^4 in the expansion of $(1 + x)(2x - 3)^5$

b Find the coefficient of x^3 in the expansion of $(x - 2)(3x + 5)^4$

9 Find, in the expansion of $\left(x^2 - \frac{1}{2x}\right)^6$, the coefficient of

a x^3 b x^6

10 Find, in the expansion of $\left(\frac{1}{t^2} + t^3\right)^{10}$, the coefficient of

a t^{10} b t^{-5}

11 The first three terms in the expansion of $(1+ax)^n$ are $1+35x+490x^2$. Given that n is a positive integer, find the value of

a n b a

12 Given that $(1+bx)^n \equiv 1-24x+252x^2+\ldots$ for a positive integer n find the value of

a n b b

13 In the expansion of $(1+2x)^n$, n a positive integer, the coefficient of x^2 is eight times the coefficient of x. Find the value of n

14 In the expansion of $\left(1+\frac{x}{2}\right)^n$, n a positive integer, the coefficients of x^4 and x^5 are equal. Calculate the value of n

15 Find an expression for

a $\binom{n}{n-1}$ b $\binom{n}{3}$

c $\binom{n}{n-2} - \binom{n+1}{n-1}$

Write your answers as polynomials in n with simplified coefficients.

16 Fully simplify these expressions.

a $\frac{n!}{(n+1)!}$ b $\frac{(n+3)!}{n(n+1)!}$

17 Find the constant term in the expansion of $(2+3x)^3\left(\frac{1}{x}-4\right)^4$

18 Find the coefficient of y^3 in the expansion of $(y+5)^3(2-y)^5$

Challenge

19 A test involves 6 questions.

For each question there is a 25% chance that a student will answer it correctly.

a How many ways are there of getting exactly two of the questions correct?

b What is the probability of getting the first two questions correct then the next four questions incorrect?

c What is the probability of getting exactly two questions correct?

d What is the probability of getting exactly half of the questions correct?

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2.3 Algebraic division

Fluency and skills

In Section **2.1** you learned how to factorise a polynomial by writing the identity and comparing and evaluating constants.

You can also use the method of dividing the polynomial by a known factor. You can divide algebraically using the same method as 'long division' in arithmetic. It is an easier method than comparing coefficients when the polynomials are of degree 3 or higher.

> **ICT Resource online**
>
> ICT resource
>
> To investigate algebraic division, click this link in the digital book.

Example 1

Use long division to divide $2x^4 + 7x^3 - 14x^2 - 3x + 15$ by $(x + 5)$

Give your answer in the form of a quotient and remainder.

$$
\begin{array}{r}
2x^3 - 3x^2 + x - 8 \\
(x+5)\overline{)\,2x^4 + 7x^3 - 14x^2 - 3x + 15} \\
\underline{2x^4 + 10x^3} \qquad\qquad\qquad\quad \\
-3x^3 - 14x^2 \qquad\qquad \\
\underline{-3x^3 - 15x^2} \qquad\qquad \\
x^2 - 3x \qquad \\
\underline{x^2 + 5x} \qquad \\
-8x + 15 \\
\underline{-8x - 40} \\
55
\end{array}
$$

Divide the first term $2x^4$ by x. Write the answer, $2x^3$, on the top.

Write $(x + 5) \times 2x^3 \equiv 2x^4 + 10x^3$ on this line and subtract from the line above to give $-3x^3$

Write the $-3x^3$ and bring down the next term, $-14x^2$, to make $-3x^3 - 14x^2$ here.

Repeat this process until you get a quotient (and a remainder if there is one).

So $(2x^4 + 7x^3 - 14x^2 - 3x + 15) \div (x + 5)$
$= (2x^3 - 3x^2 + x - 8)$ remainder 55

$(2x^3 - 3x^2 + x - 8)$ is the quotient. 55 is the remainder.

Example 2

Use long division to show that $(x - 2)$ is a factor of $f(x) = x^3 + 10x^2 + 11x - 70$

$$
\begin{array}{r}
x^2 + 12x + 35 \\
(x-2)\overline{)\,x^3 + 10x^2 + 11x - 70} \\
\underline{x^3 - 2x^2} \qquad\qquad\qquad \\
12x^2 + 11x \qquad\quad \\
\underline{12x^2 - 24x} \qquad\quad \\
35x - 70 \\
\underline{35x - 70} \\
0
\end{array}
$$

Divide the first term x^3 by x. Write the answer, x^2, on top.

Multiply x^2 by $(x - 2)$. Write the answer, $x^3 - 2x^2$, underneath and subtract from the line above.

Write the answer, $12x^2$, and bring the next term down.

Repeat the process.

There is no remainder when $f(x)$ is divided by $x - 2$ so $x - 2$ is a factor of $f(x)$

Example 1 shows that dividing f(x) by $(x-a)$ leaves you with a remainder, R

In general, for a polynomial f(x) of degree $n \geq 1$ and any constant a

$f(x) \equiv (x-a)\,g(x) + R$

Where g(x) is a polynomial of order $n-1$ and R is a constant.

For the particular case when $x = a$, this gives

$f(a) = (a-a)\,g(a) + R$

$f(a) = R$

You can see from this that $f(a) = 0$ implies there is no remainder when f(x) is divided by $(x-a)$

This calculation also demonstrates the **remainder theorem** which is beyond the scope of your A level course.

The **factor theorem** states that if $f(a) = 0$, $(x-a)$ is a factor of f(x) **Key point**

In Example 2 you saw that there is no remainder when $x^3 + 10x^2 + 11x - 70$ is divided by $(x-2)$, which is equivalent to saying that $(x-2)$ is a factor of $x^3 + 10x^2 + 11x - 70$

If you substitute $x = 2$ into the expression, the factor $(x-2)$ is zero so the value of f(x) is zero.

You can check this by substitution, which gives $f(2) = 2^3 + 10(2)^2 + 11(2) - 70$

$= 8 + 40 + 22 - 70 = 0$

Example 3

Show that $(x+3)$ is a factor of $2x^4 + 2x^3 - 9x^2 - 4x - 39$

$f(-3) = 2(-3)^4 + 2(-3)^3 - 9(-3)^2 - 4(-3) - 39$
$= 0$

$(x+3)$ is a factor since $f(-3) = 0$

$(x-a)$ is a factor if $f(a) = 0$, so to show $(x+3)$ is a factor you need to show that $f(-3) = 0$

Example 4

Fully factorise the polynomial $2x^3 + 17x^2 - 13x - 168$

$f(x) = 2x^3 + 17x^2 - 13x - 168$

$f(1) = 2(1)^3 + 17(1)^2 - 13(1) - 168 = -162$

$f(1) \neq 0$ so $(x-1)$ is not a factor

Use trial and error with different values of a to find a case where $f(a) = 0$

$f(2) = 2(2)^3 + 17(2)^2 - 13(2) - 168 = -110$

$f(2) \neq 0$ so $(x-2)$ is not a factor

$f(3) = 2(3)^3 + 17(3)^2 - 13(3) - 168 = 0$

$f(3) = 0$ so $(x-3)$ is a factor

$$
\begin{array}{r}
2x^2 + 23x + 56 \\
(x-3)\overline{)\,2x^3 + 17x^2 - 13x - 168} \\
\underline{2x^3 - 6x^2 \qquad\qquad\quad} \\
23x^2 - 13x \qquad \\
\underline{23x^2 - 69x \qquad} \\
56x - 168 \\
\underline{56x - 168} \\
0
\end{array}
$$

Use long division to divide the polynomial by the factor to get a quadratic expression in x

So $2x^3 + 17x^2 - 13x - 168 \equiv (x-3)(2x^2 + 23x + 56)$

$\equiv (x-3)(2x+7)(x+8)$

Use the result from the long division to express the polynomial in a partially factorised form.

Factorise the quadratic to fully factorise the polynomial.

Exercise 2.3A Fluency and skills

1 Divide

a $x^2 - x - 90$ by $(x + 9)$

b $3x^2 - 19x - 14$ by $(x - 7)$

c $8x^2 + 14x - 15$ by $(2x + 5)$

2 Divide each polynomial by the given factor by comparing coefficients.

a $x^3 + 3x^2 - 11x + 7$ by $(x - 1)$

b $x^3 + 2x^2 - 4x - 3$ by $(x + 3)$

c $2x^3 + 9x^2 - 17x - 45$ by $(2x - 5)$

d $3x^3 - 14x^2 + 16x + 7$ by $(3x + 1)$

e $2x^4 - 17x^3 + 22x^2 + 65x - 9$ by $(2x - 9)$

3 Use long division to divide $4x^3 + 4x^2 - 8x + 5$ by $(x - 4)$

4 Use long division to show that $5x^3 + 11x^2 - 73x - 15$ is divisible by $(x - 3)$

5 Divide using long division

a $x^3 - 2x + 1$ by $(x - 1)$

b $x^3 - 10x^2 - 10x - 11$ by $(x - 11)$

c $6x^3 - 13x^2 - 19x + 12$ by $(3x + 4)$

d $6x^4 - 19x^3 + 23x^2 - 26x + 21$ by $(2x - 3)$

e $10x^4 + 33x^3 - 57x^2 + 5x + 1$ by $(5x - 1)$

6 Work out the values of **i** f(0) **ii** f(1) **iii** f(–1) **iv** f(2) **v** f(–2) when

a $f(x) = x^3 - 2x^2 + 10x$

b $f(x) = x^3 - 2x^2 - 2x - 2$

c $f(x) = x^3 - 3x^2 + x + 2$

d $f(x) = 2x^3 + x^2 - 5x + 2$

e $f(x) = x^3 - x^2 - 4x + 4$

7 a Show that $(x + 6)$ is a factor of $x^3 + 4x^2 - 9x + 18$

b Show that $(x - 8)$ is a factor of $2x^3 - 13x^2 - 20x - 32$

c Show that $(3x - 1)$ is a factor of $3x^3 + 11x^2 - 25x + 7$

d Show that $(5x + 2)$ is a factor of $10x^3 + 19x^2 - 39x - 18$

8 Fully factorise the polynomial $4x^3 + 27x^2 - 7x$

9 Fully factorise the polynomial $2x^3 + 9x^2 - 2x - 9$

10 a Factorise fully $x^3 + 3x^2 - 16x + 12$

b Factorise fully $x^3 - 6x^2 - 55x + 252$

c Factorise fully $6x^3 + 19x^2 + x - 6$

d Factorise fully $x^4 - 13x^2 - 48$

Reasoning and problem-solving

Strategy

To factorise a polynomial

1. Apply the factor theorem as necessary to find your first factor.
2. Divide the polynomial by the factor to get a quadratic quotient.
3. Factorise the quadratic quotient to fully factorise the polynomial.

Example 5

$(x + 1)$ is a factor of the polynomial $3x^3 + 8x^2 + ax - 28$. Fully factorise the polynomial.

$f(-1) = 0$ — ① Use the factor theorem, f(*a*) = 0, to form an expression in *a*

$\Rightarrow 3(-1)^3 + 8(-1)^2 + a(-1) - 28 = 0$

$-3 + 8 - a - 28 = 0 \Rightarrow a = -23$ — Simplify to find the value of *a*

$$\begin{array}{r} 3x^2 + 5x - 28 \\ (x+1)\overline{)\,3x^3 + 8x^2 - 23x - 28} \end{array}$$

② Use long division to get a quadratic quotient (the full calculation isn't shown here).

$(x + 1)(3x^2 + 5x - 28) \equiv (x + 1)(3x - 7)(x + 4)$ — ③ Factorise the quadratic

Exercise 2.3B Reasoning and problem-solving

1 **a** $2x^4 + px^3 - 6x^2 + qx + 6$ is divisible by $(x-1)$

Use this information to write an equation in p and q

b $2x^4 + px^3 - 6x^2 + qx + 6$ is divisible by $(x+3)$

Use this information to write an equation in p and q

c Solve these equations simultaneously to find the values of p and q

2 **a** Work out the value of a when $2x^3 + ax^2 - 4x + 1$ is divisible by $(x-2)$

b Work out the value of b when $x^4 + (b^2+1)x^3 + bx^2 + 7x - 15$ is divisible by both $(x+5)$ and $(x-1)$

c Work out the values of p and q when $2x^4 + px^3 - 6x^2 + qx + 6$ is divisible by $(x^2 + 2x - 3)$

3 $x^3 - 4x^2 - 31x + 70$; $x^2 + 3x - 10$ and $x^2 - 9x + 14$ have one common factor. What is it?

4 What is the LCM of $x^2 + 4x + 3$ and $x^2 + x - 6$?

5 What is the highest common factor of $x^3 + 4x^2 + x - 6$ and $x^3 + 3x^2 - x - 3$?

6 Find the LCM and HCF of $2x^2 + x - 21$ and $2x^2 + 15x + 28$

7 Find the LCM and HCF of $x^3 + 7x^2 - 53x - 315$ and $x^3 + 21x^2 + 143x + 315$

8 $f(x) = x^3 + 9x^2 + 11x - 21$ and $g(x) = x^3 + 2x^2 - 13x + 10$

Find the common factor of $f(x)$ and $g(x)$ and show that it is also a factor of $f(x) - g(x)$

9 Find the values of a and b if $5x - 4$ and $x + 3$ are factors of $ax^2 + 33x + b$

10 **a** A circle's area is $\pi(4x^2 - 12x + 9)$ m^2. Work out its radius.

b The volume of a square-based pyramid is $(2x^3 - 5x^2 - 24x + 63)$ cm^3. The height is $(2x + 7)$ cm. Work out the length of the side of the square base.

11 **a** The velocity of a moving body is $2t^3 - 19t^2 + 57t - 54$ m s^{-1} at any time t. When is the body stationary?

b The acceleration of the same body is $6t^2 - 38t + 57$ m s^{-2}. Work out

i The acceleration of the body when the velocities are zero,

ii The exact times when the acceleration is zero.

12 The volume of a cone is $\frac{\pi}{3}(3x^3 - 11x^2 - 15x + 63)$ m^3

a Work out possible values of the radius and height of the cone in terms of x

b What is the range of possible values of x?

13 Part of a rollercoaster ride is modelled by the equation $h = t^3 - 12t^2 + 41t - 30$ where h is the height above ground level in metres and t is the time in seconds. Work out

a At what times the ride is at ground level,

b When, between these times, the ride is above the ground level.

14 **a** A pyramid has a rectangular base. Its volume is given by $V = x^3 + 7x^2 + 14x + 8$ cm^3 Work out the possible values for its dimensions.

b What is the range of possible values of x?

Challenge

15 A sphere, radius $(x + 5)$ cm, has a concentric sphere, radius $(x - 3)$ cm removed. Use the identity $A^3 - B^3 \equiv (A - B)(A^2 + AB + B^2)$ to work out the volume of the shell. Give the volume in expanded form.

2.4 Curve sketching

Fluency and skills

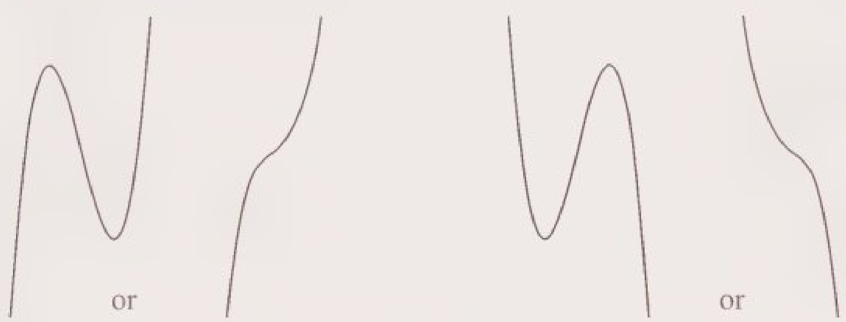
You can sketch the graph of a function without plotting a large number of points. A sketch should show the key features of a function.

- Its general shape including any symmetry,
- Its x- and y-intercepts.

Many sketches also show maximum points, minimum points and points of inflection.

A **cubic** function can be written in the form $y = ax^3 + bx^2 + cx + d$, where a, b, c and d are constants and $a \neq 0$

Cubic curves take the form



$y = ax^3 + bx^2 + cx + d,\ a > 0$

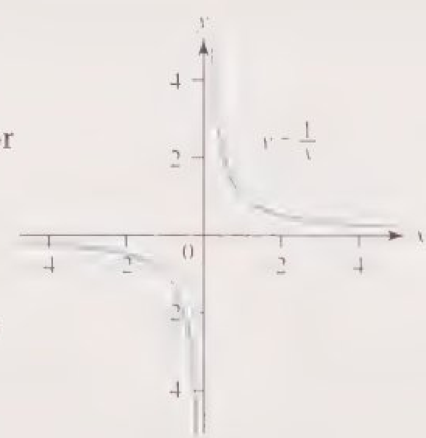
$y = ax^3 + bx^2 + cx + d,\ a < 0$

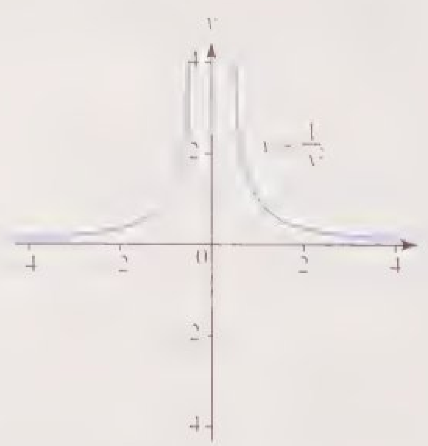
> You can use a graphics calculator to sketch curves.

> At a point of inflection the concavity of the curve changes: it bends in the other direction.

Reciprocal curves such as $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ exhibit interesting behaviour as they are undefined for certain values of x

Both of these reciprocal functions are undefined for $x = 0$ as you cannot divide by 0. As x gets closer to 0, y approaches infinity, ∞, or negative infinity, $-\infty$





In both of these functions, as the magnitude of x gets bigger and bigger, y gets increasingly close to zero but never reaches zero.

The x- and y-axes are **asymptotes** to the curve in each case.

> **Key point**
> A line, l, is an asymptote to a curve, C, if, along some unbounded section of the curve, the distance between C and l approaches zero.

Example 1

a For a constant $a > 1$ sketch these curves on one set of axes.

i $f(x) = (a - x)(x + 1)(x + 2a)$ **ii** $g(x) = \dfrac{2}{x - a}$

b Show that there are no positive solutions to the equation $-(a - x)^2(x + 1)(x + 2a) - 2 = 0$

a i x-intercepts: $a - x = 0 \Rightarrow x = a$

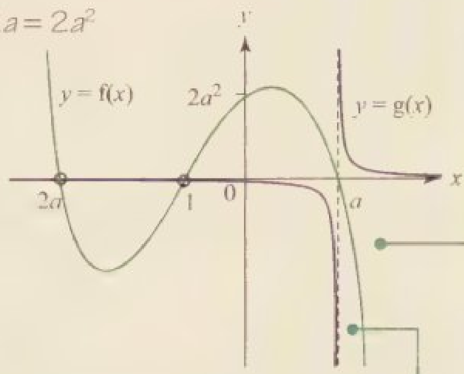
$x + 1 = 0 \Rightarrow x = -1$

$x + 2a = 0 \Rightarrow x = -2a$

y-intercept: $x = 0 \Rightarrow y = a \times 1 \times 2a = 2a^2$

The coefficient of x^3 is $-1 \times 1 \times 1 = -1 < 0$

a ii Undefined when $x - a = 0 \Rightarrow x = a$



As the magnitude of x gets bigger and bigger the value of y gets closer to 0

Negative cubic shape, $a > 1$ and $-2a < -1$

You cannot divide by 0 so as x gets closer to a, y gets closer to ∞ or $-\infty$

b $-(a - x)^2(x + 1)(x + 2a) - 2 = 0$

$(a - x)(x + 1)(x + 2a) = \dfrac{2}{-(a - x)}$

$(a - x)(x + 1)(x + 2a) = \dfrac{2}{(x - a)} \Rightarrow f(x) = g(x)$

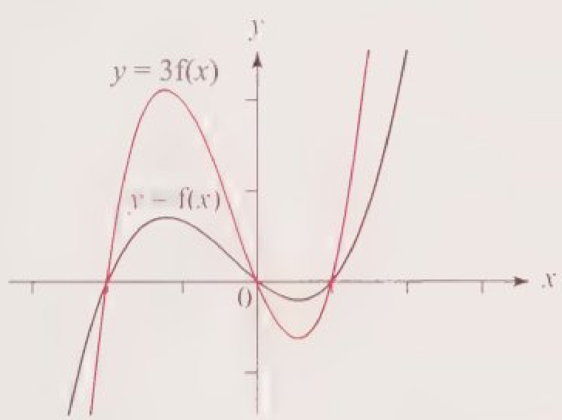
The equation is satisfied at the points of intersection of f(x) and g(x). From the graph, the curves have two points of intersection and both have negative x-coordinates, so there are no positive solutions.

PURE

Transformations can help you to see how different functions relate to one another.

You will work with four common transformations in this chapter.

y = af(x) is a vertical stretch of $y = f(x)$ with scale factor a



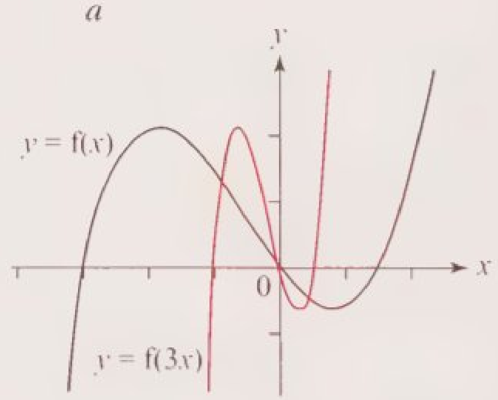
In the transformation $y = af(x)$, the x-values remain unchanged and each y-value is multiplied by a

Every point $(x, f(x))$ becomes $(x, af(x))$

Key point

If $a < 0$ the transformation $y = af(x)$ reflects the curve in the x-axis.

y = f(ax) is a horizontal stretch of $y = f(x)$ with scale factor $\dfrac{1}{a}$



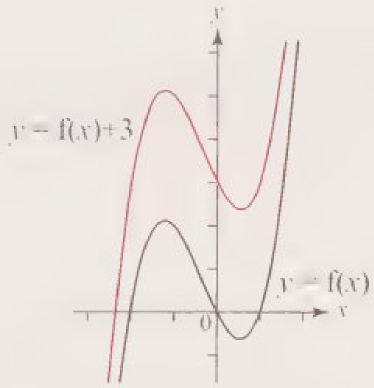
In the transformation $y = f(ax)$, each x-value is multiplied by a before the corresponding y-value is calculated.

Every point $(x, f(x))$ becomes $(x, f(ax))$

Key point

If $a < 0$ the transformation $y = f(ax)$ reflects the curve in the y-axis. If $-1 < a < 1$ the curve gets wider.

$y = f(x) + a$ is a translation of $y = f(x)$ by the vector $\begin{pmatrix} 0 \\ a \end{pmatrix}$



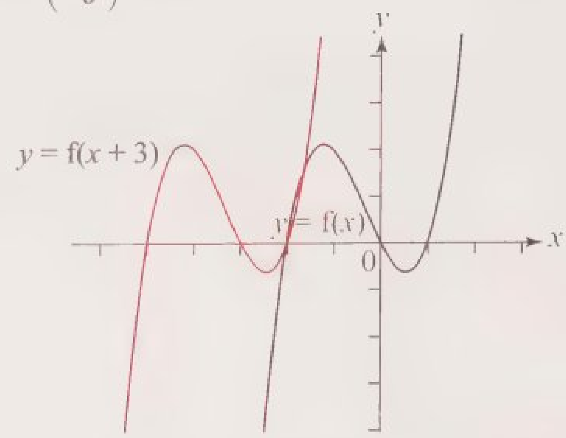
In the transformation $y = f(x) + a$, the x-values remain unchanged and each y-value is increased by a

Every point $(x, f(x))$ becomes $(x, f(x) + a)$

Key point

If $a < 0$ the transformation $y = f(x) + a$ translates the curve downwards.

$y = f(x + a)$ is a translation of $y = f(x)$ by the vector $\begin{pmatrix} -a \\ 0 \end{pmatrix}$



In the transformation $y = f(x + a)$, a is added to each x-value before the corresponding y-value is calculated.

Every point $(x, f(x))$ becomes $(x, f(x + a))$

Key point

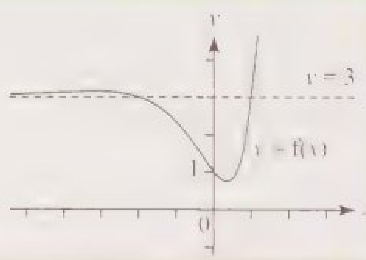
If $a < 0$ the transformation $y = f(x + a)$ moves the curve to the right.

Example 2

The graph shows a sketch of the curve $y = f(x)$

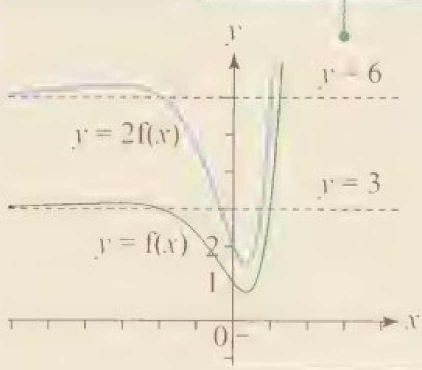
Sketch the curves

a $y = 2f(x)$ **b** $y = f(x) - 1$ **c** $y = f(x - 1)$ **d** $y = f(-x)$

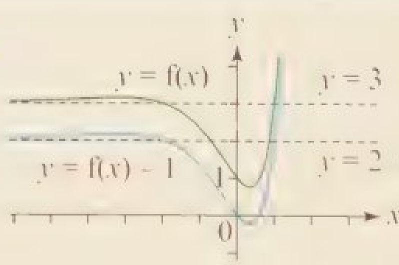


The translated curve has asymptote $y = 2 \times 3 = 6$

a



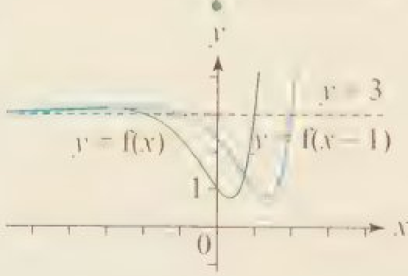
b



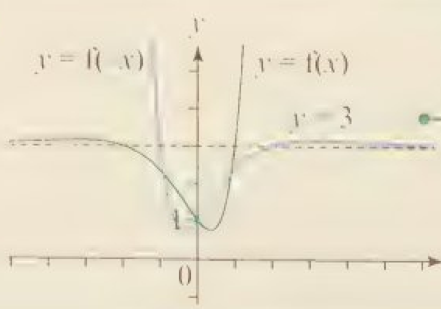
The translated curve has asymptote $y = 3 - 1 = 2$

You don't have enough information to mark the y-intercept.

c



d



The curve is reflected in the y-axis.

Exercise 2.4A Fluency and skills

1 Evaluate all the x-intercepts for these graphs. Show your working.

a $y=x^2-x-6$ b $y=2x^2-9x-35$

c $y=x^3+8$ d $y=2x^3-54$

e $y=(x-3)^4$ f $y=(2x+5)^4-7$

2 Identify all the vertical and horizontal asymptotes for $y=\frac{3}{x-1}$. Show your working.

3 Evaluate all axes of symmetry in these graphs. Show your working.

a $y=x^2-8x-9$ b $y=(x+2)^4$

c $(y-3)^2=x+4$ d $y=(x-4)^2(x+3)^2$

Hence sketch the graph of each function.

4 Sketch the graphs of these functions.

a $y=x^3+3$ b $y=(x-3)^3$

c $y=-2x^3+3$ d $y=2(x+3)^3-1$

e $y=(2x+1)^3$ f $y=5+(3x-4)^3$

g $y=x^3-5x^2-14x$

h $y=(x+5)(x-6)(2x+1)$

i $y=\frac{-2}{x}$ j $y=\frac{4}{x+2}$

k $y=\frac{-5}{x-7}$

5 Sketch the graphs of these functions.

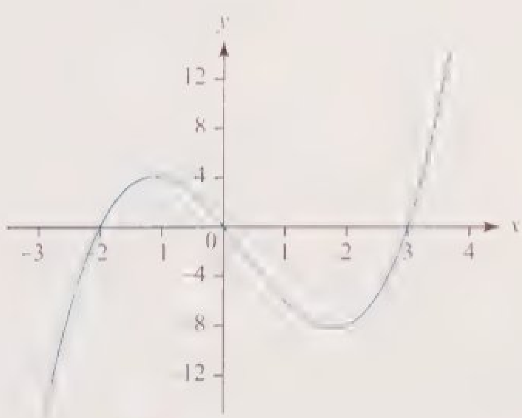
a $y=5x^3-2x^4$ b $y=5x^2+2x^3$

c $y=x^3-3x^2$ d $y=(1-x)(x+3)^2$

e $y=x^2(x-3)^2$ f $y=x^2(x+3)^2$

g $y=x^4-7x^3$ h $y=(x^2-4)(x^2-9)$

6 The graph of $y=f(x)$ is shown.



Sketch the graphs of

a $y=f(2x)$ b $y=f(x-2)$

c $y=f\left(\frac{x}{3}\right)$ d $y=f(x+3)$

e $y=f(-x)$ f $y=-f(x)$

7 The graph of $y=g(x)$ has a maximum point at $(-2, 5)$ and a minimum point at $(8, -4)$
State the coordinates of the maximum and minimum points of these transformed graphs.

a $y=g(4x)$ b $y=3g(x)$

c $y=g(x+7)$ d $y=g(x)+4$

e $y=\frac{1}{2}g(x)$ f $y=-g(x)$

g $y=g(-x)$ h $y=g\left(\frac{x}{2}\right)$

8 Describe each of the transformations in question 7

9 $f(x)=x^3$. Write down the equation when the graph of $y=f(x)$ is

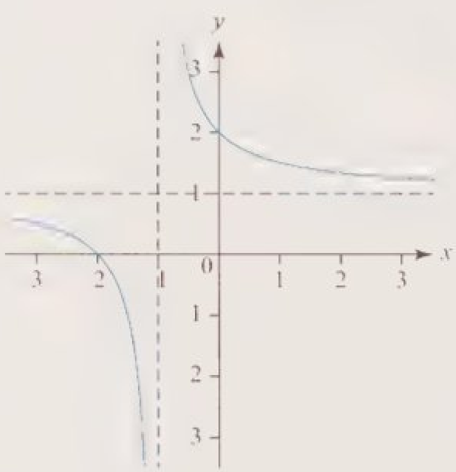
a Translated 3 units left,

b Translated 2 units up,

c Stretched vertically by scale factor 2,

d Stretched horizontally by scale factor 3

10 The graph of $y=f(x)$ is shown.



Sketch the graphs of

a $y=f(x+3)$ b $y=3f(x)$

c $y=f\left(\frac{x}{2}\right)$ d $y=f(x)+1$

Reasoning and problem-solving

Strategy

When sketching a graph

1. Define the function using any variables supplied in the question.
2. Identify the standard shape of the curve and identify any symmetry.
3. Identify any x-and y-intercepts and any asymptotes.
4. Apply any suitable transformations.
5. Show all relevant information on your sketch.

You can use graphs to show proportional relationships.

If y is proportional to x, you write $y \propto x$. This can be converted to an equation using a constant of proportionality, giving $y = kx$. The graph of y against x is a straight line through the origin with gradient k

If y is inversely proportional to x you write $y \propto \frac{1}{x}$ or $y = \frac{k}{x}$. The graph of $y = \frac{k}{x}$ is a vertical stretch, scale factor k, of the graph $y = \frac{1}{x}$

Example 3

A rectangle has a fixed area of 36 m². Its length, y m is inversely proportional to its width, x m.

a Write a formula for y in terms of x

b Without plotting exact points, sketch the graph of your function.

c Explain any asymptotes that the graph has.

a $y \propto \frac{1}{x}$ so $y = \frac{k}{x}$ — (1) y is inversely proportional to x

$xy = 36$ so $k = 36$ — The area is fixed at 36 m².

$y = \frac{36}{x}$

b $y = \frac{36}{x}$

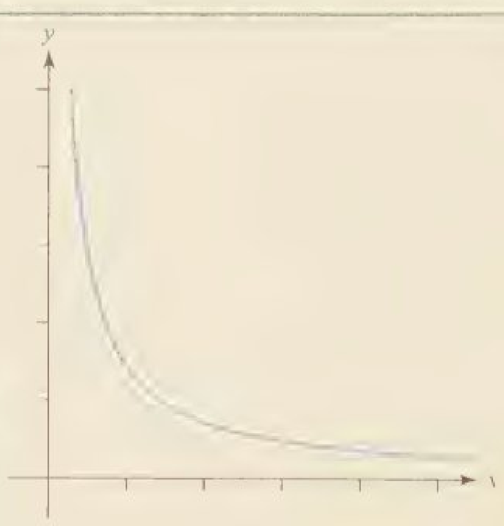
When $x = 0$, y is not defined.

The line $x = 0$ is an asymptote.

$x = \frac{36}{y}$

When $y = 0$, x is not defined.

The line $y = 0$ is an asymptote.



(2)(3)(5) Apply what you know about graphs of the form $y = \frac{k}{x}$

c y and x are actual lengths, so they must be positive and the curve approaches the asymptotes as shown.

Exercise 2.4B Reasoning and problem-solving

1 The radius, r, of a container is inversely proportional to its height, h

A container of radius 4 cm will have a height of 14 cm.

  a Write an equation linking h and r

  b Sketch a graph to illustrate this relationship.

2 The volume, $v\text{ cm}^3$ of water in a tank is proportional to the square-root of the time, t seconds. After 15 minutes the tank has 1800 cm^3 of water in it.

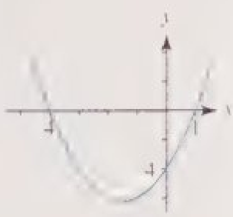
  a Write an equation linking v and t

  b Sketch a graph to illustrate this relationship.

3 a Sketch the graphs of $y=\dfrac{1}{x+2}$ and $y=x^2(x-3)$ on the same axes.

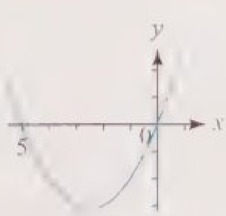
  b Use your answer to part **a** to explain how many solutions there are to the equation $x^2(x-3)=\dfrac{1}{x+2}$

4 The graph of $y=\mathrm{f}(x)$ is shown.



Give the equations for each of these transformations in terms of $\mathrm{f}(x)$

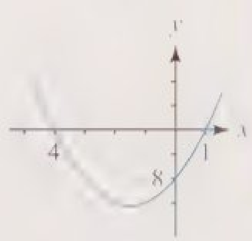
a



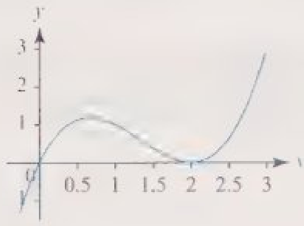
b

c

d

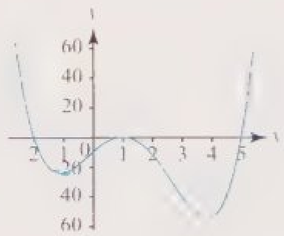


5 The graph of $y=x^3+Ax^2+Bx+C$ is shown.



Find the values of the constants A, B and C

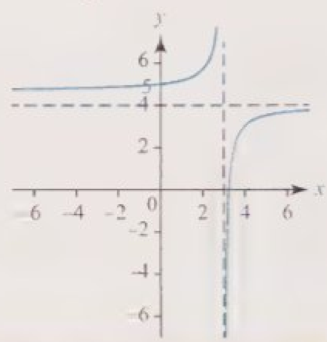
6 This is the graph of $y=\mathrm{f}(x)$ where $\mathrm{f}(x)=x^4+Ax^3+Bx^2+Cx-10$



Find the values of the constants A, B, and C

7 The graph shown has the equation $y=A+\dfrac{B}{x+C}$

Find the values of A, B and C



Challenge

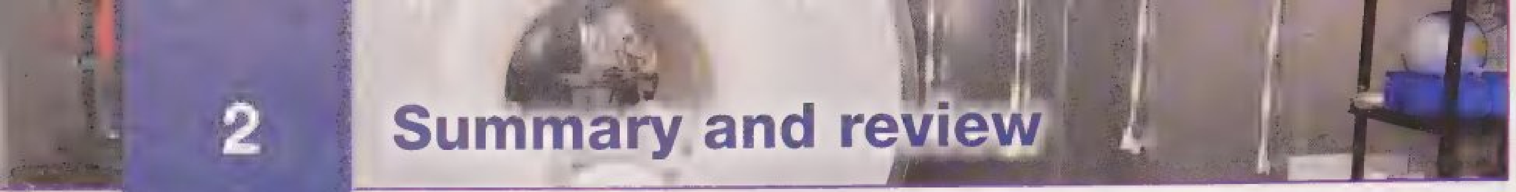
8 For the graph of $y=ax^2+bx+c$ where a, b and c are constants

  a Explain the conditions for the graph to have a minimum point and the conditions for the graph to have a maximum point,

  b Write down the coordinates of the maximum or minimum point,

  c Write down the coordinates where the curve intersects the axes,

  d Write down the equation of the line of symmetry of the curve.



2 Summary and review

Chapter summary

- The highest power in a polynomial expression is called its degree.
- When adding or subtracting polynomials, expand brackets before collecting like terms.
- Identities use the ≡ sign. Identities are true for all values of the variable(s).
- For $n = 0, 1, 2, 3, \ldots$, the binomial expansions are

$$(1+x)^n \equiv 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots+x^n$$

and

$$(a+b)^n \equiv a^n + {}^nC_1a^{n-1}b + {}^nC_2a^{n-2}b^2 + \ldots + {}^nC_ra^{n-r}b^r + \ldots + b^n$$

- The coefficients of these expansions can be found from Pascal's triangle or from ${}^nC_r \equiv \frac{n!}{(n-r)!r!}$
- You can divide algebraically using the same technique as for long division in arithmetic.
- The factor theorem states that if $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$
- To sketch a graph you need to consider the symmetry, x- and y-intercepts, asymptotes, behaviour as x and/or y approaches $\pm\infty$, and any other obvious critical points. You can also use your knowledge of transformations.

Check and review

You should now be able toô	Try Questions
✔ Manipulate, simplify and factorise polynomials.	1–4, 17
✔ Understand, expand and use the binomial theorem.	7–11
✔ Divide polynomials by algebraic expressions.	6, 12, 14, 15
✔ Understand and use the factor theorem.	5, 13, 16
✔ Use a variety of techniques to analyse a function and sketch its graph.	18–23

1 Add together $2x^4 + 9x^5 + 11x^2 - 3x - 5x^4 - 12$ and $4x^2 - x^4 - 7x^5 + 3 + 12x - 5x^3$

2 Fully factorise $4n^3 + 4n^2 - 15n$

3 Expand and simplify these expressions.

a $(y-1)(y+3)(2y+5)$

b $(2z+1)(z-2)^2$

4 Factorise these expressions.

a $m(m+4)-(m+4)^2$

b $(d+1)^2-4(d+1)(d-1)$

5 The equation $2x^3 + ax^2 + bx + c = 0$ has roots -4, 3 and $\frac{7}{2}$. Find the values of a, b and c

6 Find the function that, when divided by $(x+3)$, gives a quotient of $(2x-3)$ and a remainder of -4

7 Use Pascal's triangle to write the expansion of $\left(1+\frac{m}{10}\right)^4$

Use your answer to evaluate the value of 1.1^4 to 4 decimal places.

8 Use the binomial theorem to expand $(2s^2-4t)^4$

9 Use Pascal's triangle to expand and simplify these expressions.

a $(1+\sqrt{3})^4$ b $(3-\sqrt{5})^5-(3+\sqrt{5})^5$

10 Find the constant term in the binomial expansion of $\left(w-\frac{3}{2w}\right)^{14}$

11 a $(2-ax)^9 \equiv 512+2304x+bx^2+cx^3+\ldots$
Find the values of a, b, and c

b Use your values of a, b, and c to find the first four terms in the expansion of $(1-x)(2-ax)^9$

12 Divide $2x^3-3x^2-26x+3$ by $(x+3)$

13 By successively evaluating f(1), f(−1), f(2), f(−2) and so on, find all the factors of x^3-4x^2+x+6

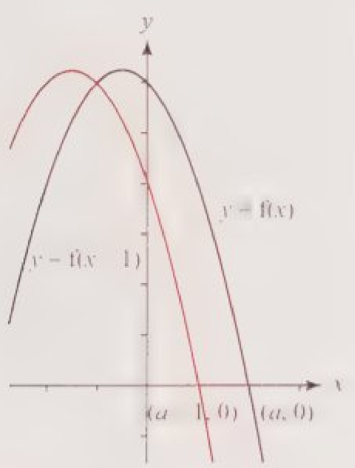
14 Divide $8x^3+14x^2-x+35$ by $(2x+5)$

15 Divide x^3-2x^2+3x+4 by $(x-2)$

16 Show that $(2x-3)$ is a factor of $4x^3-8x^2+x+3$

17 Factorise fully $2x^3+x^2-18x-9$

18 Susan attempted to transform the graph of $y=f(x)$ into $y=f(x-1)$



a Explain what mistake she has made.

b Sketch the graph of $y=f(x-1)$

19 Sketch these curves on the same set of axes.

a $y=\frac{1}{x}$ b $y=\frac{4}{x}$ c $y=2+\frac{1}{x}$

20 Sketch the graph of $y=(x-6)^3$

21 A particle moves along a straight line from O, so that, at time t s, it is s m from O, given by the equation $s=t(2t-7)^2$

Sketch the graph and describe its motion fully.

22 A rectangular metal sheet, 16 in by 10 in, has squares of side x in removed from its corners.

The edges are turned up to form an open box.

a Show that the volume of this box is $V=160x-52x^2+4x^3$ in^3

b Sketch a graph to evaluate the value of x that gives the highest volume.

23 A particle moves along a straight line from O, so that, at time t seconds, it is s metres from O, given by the equation $s=t(t-4)^2$. Sketch the graph and describe its motion.

What next?

Score		
	0 – 11	Your knowledge of this topic is still developing. To improve, search in MyMaths for the codes: 2006, 2022–2024, 2027, 2041–2043, 2258
	12 – 17	You're gaining a secure knowledge of this topic. To improve, look at the InvisiPen videos for Fluency and skills (02A)
	18 – 23	You've mastered these skills. Well done, you're ready to progress! To develop your techniques, look at the InvisiPen videos for Reasoning and problem-solving (02B)

Click these links in the digital book

Exploration

Going beyond the exams

History

The binomial theorem is a formula for finding any power of a two-term bracket without having to multiply them all out. It has existed in various forms for centuries and special cases, for low powers, were known in Ancient Greece, India and Persia.

The triangular arrangement of the binomial coefficients is known as **Pascal's triangle**. It took its name from the 17th century mathematician **Blaise Pascal**, who studied its properties in great depth. Although the triangle is named after Pascal, it had been known about much earlier. A proof linking it to the binomial theorem was given by an Iranian mathematician Al-Karaji in the 11th century.

Around 1665, **Isaac Newton** developed the binomial theorem further by applying it to powers other than positive whole numbers. He showed that a general formula worked with any rational value, positive or negative.

Newton showed how the binomial theorem could be used to simplify the calculation of roots and also used it in a calculation of π, which he found to 16 decimal places.

Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Have a go

For small values of x, $(1 + x)^n \approx 1 + nx$

Use this result to estimate the value of

a) $(1.02)^4$

b) $(0.99)^5$

c) $(2.01)^5$

Find these values on a calculator and compare your results.

$(2.01)^5$ can be written in the form $2^5(1 + ...)^5$

"If I have seen further than others, it is by standing on the shoulders of giants."
- Isaac Newton

2 Assessment

1 Factorise $2s^4 + 2s^3 - 12s^2$ Choose the correct answer.

A $(s+3)(s-2)$ B $s^2(s+3)(s-2)$ C $2(s^2+3)(s^2-2)$ D $2s^2(s+3)(s-2)$ **[1 mark]**

2 Simplify $\frac{3(x+1)-1}{2x^2+x-3} - \frac{1}{x-1}$ Choose the correct answer.

A $\frac{x+5}{(2x+3)(x-1)}$ B $\frac{1}{2x+3}$ C $\frac{3x+2}{x-1}$ D $\frac{1}{2x+1}$ **[1]**

3 a Simplify these expressions.

i $(2x-3)(6x+1)$ ii $(2a-3b)^2$ iii $(5x+2y)(x^2-3xy-y^2)$ **[6]**

b Given $\frac{ax^2+bx+c}{(3x+4)} \equiv (3x-4)$, evaluate the values of the constants a and c, and show that $b=0$ **[3]**

4 Write down the binomial expansion of $\left(1+\frac{1}{2}x\right)^8$ in ascending powers of x, up to and including the term in x^3. Simplify the terms as much as possible. **[6]**

5 a Factorise p^3-10p^2+25p **[2]**

b Deduce that $(2x+5)^3-10(2x+5)^2+25(2x+5) \equiv ax^2(2x+5)$, where a is a constant that should be stated. **[2]**

6 Show that $(x-3)$ is *not* a factor of $2x^3-5x^2+6x-7$ **[3]**

7 Show how the binomial expansion can be used to work out each of these without a calculator.

a 268^2-232^2 **[2]**

b $469 \times 548 + 469^2 - 469 \times 17$ **[2]**

c $\frac{65.1 \times 29.2 + 65.1 \times 35.9 - 91.7 \times 26.4 + 65.3 \times 26.4}{18.3^2 - 18.3 \times 5.4}$ **[5]**

8 Given that $(1+cx)^7 \equiv 1+21x+Ax^2+Bx^3+\ldots\ldots\ldots,$

a Work out i The value of c ii The value of A iii The value of B **[4]**

b Using your values of c, A and B, evaluate the coefficient of x^3 in the expansion of $(2+x)(1+cx)^7$ **[2]**

9 Express x^3-3x^2+5x+1 in the form $(x-2)(x^2+ax+b)+c$ **[3]**

10 a Write down the expansions of i $(x+y)^4$ ii $(x-y)^4$ **[4]**

b Show that $(\sqrt{5}+\sqrt{2})^4+(\sqrt{5}-\sqrt{2})^4=n$, where n is an integer to be found. **[4]**

11 Write down the term which is independent of x in the expansion of $\left(x^2+\frac{2}{x}\right)^9$ **[3]**

12 a Expand each of these in ascending powers of x up to and including the term in x^2

i $(1+2x)^6$ ii $(2-x)^6$ **[5]**

b Hence write down the first three terms in the binomial expansion of $(2+3x-2x^2)^6$ **[4]**

13 a Show that $(x-2)$ is a factor of $2x^3+x^2-7x-6$ **[2]**

b Show that the equation $2x^3+x^2-7x-6=0$ has the solutions 2, $-\frac{3}{2}$, and -1 **[5]**

14 Given that both $(x-1)$ and $(x+3)$ are factors of $ax^3+bx^2-16x+15$

a Evaluate the values of a and b **[6]**

b Fully factorise $ax^3+bx^2-16x+15$ **[3]**

c Sketch the graph of $y=ax^3+bx^2-16x+15$ **[3]**

d Solve the inequality $ax^3+bx^2-16x+15\geq 0$ **[2]**

15 a Expand $\left(x+\frac{1}{x}\right)^6$, simplifying the terms. **[7]**

b Hence write down the expansion of $\left(x-\frac{1}{x}\right)^6$ **[1]**

c Prove that the equation $\left(x+\frac{1}{x}\right)^6-\left(x-\frac{1}{x}\right)^6=64$ has precisely two real solutions. **[5]**

16 Prove these results

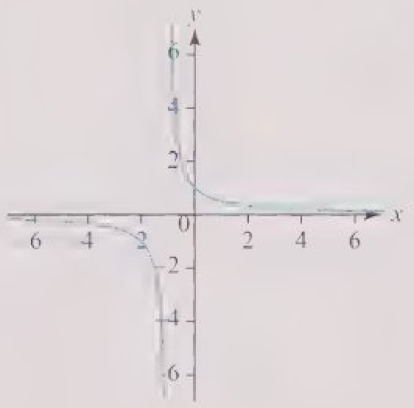
a ${}^{n+1}C_r \equiv {}^{n}C_r + {}^{n}C_{r-1}$ **[6]**

b ${}^{n+2}C_3 - {}^{n}C_3 \equiv n^2$ **[8]**

17 Here are five equations, labelled **i – v**, and five graphs, labelled **A – E**

i $y=\frac{1}{(x-2)^2}$ **ii** $y=1+\frac{1}{(x-1)}$ **iii** $y=-\frac{1}{x+1}$ **iv** $y=-\frac{1}{(x+1)^2}$ **v** $y=1+\frac{1}{x^2}$

A



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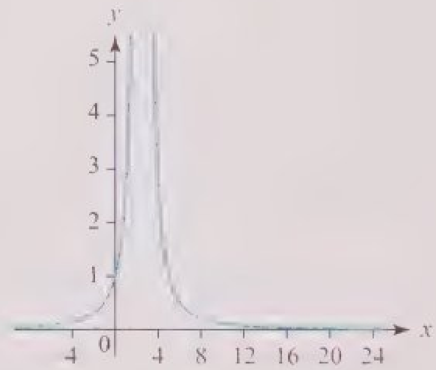

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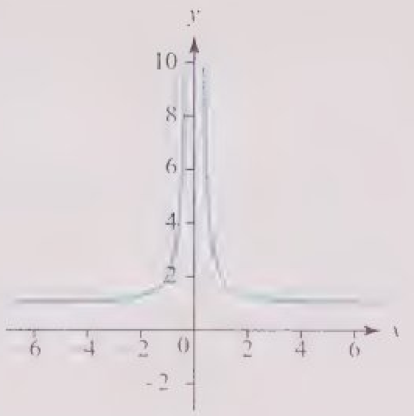
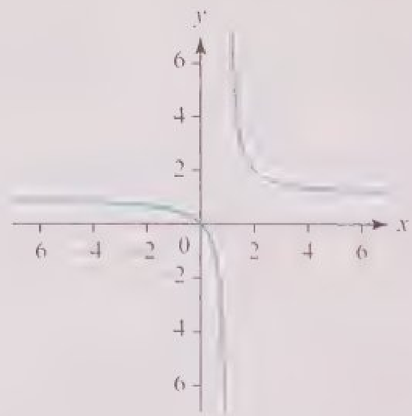


D



E

Four of the equations correspond to four of the graphs.

a Match the four equations to their graphs. **[4]**

b For the graph that has no equation, write down a possible equation. **[1]**

c For the equation that has no graph, sketch its graph. **[3]**