AQA
A Level Maths

Year 1 / AS Level

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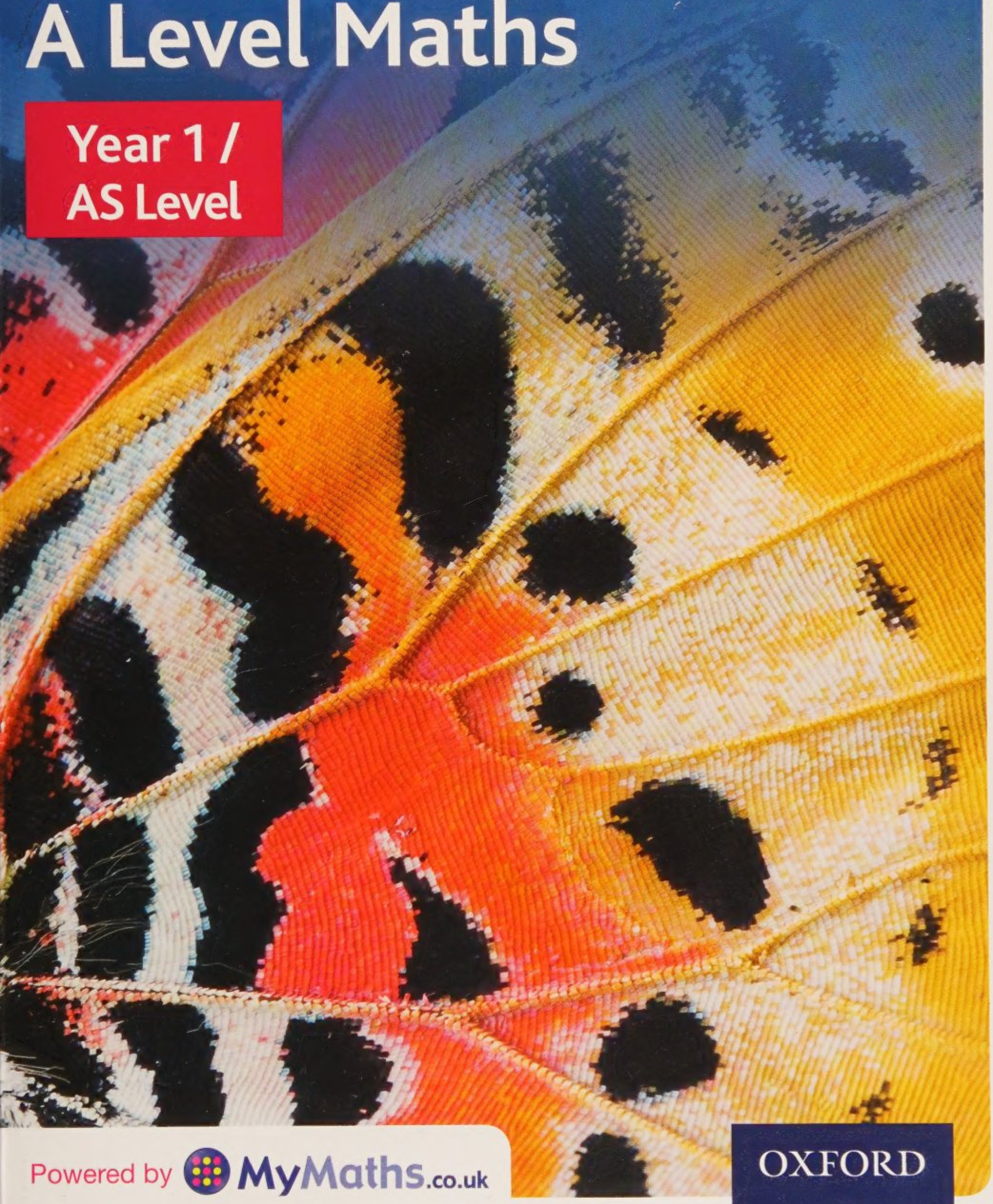

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OXFORD

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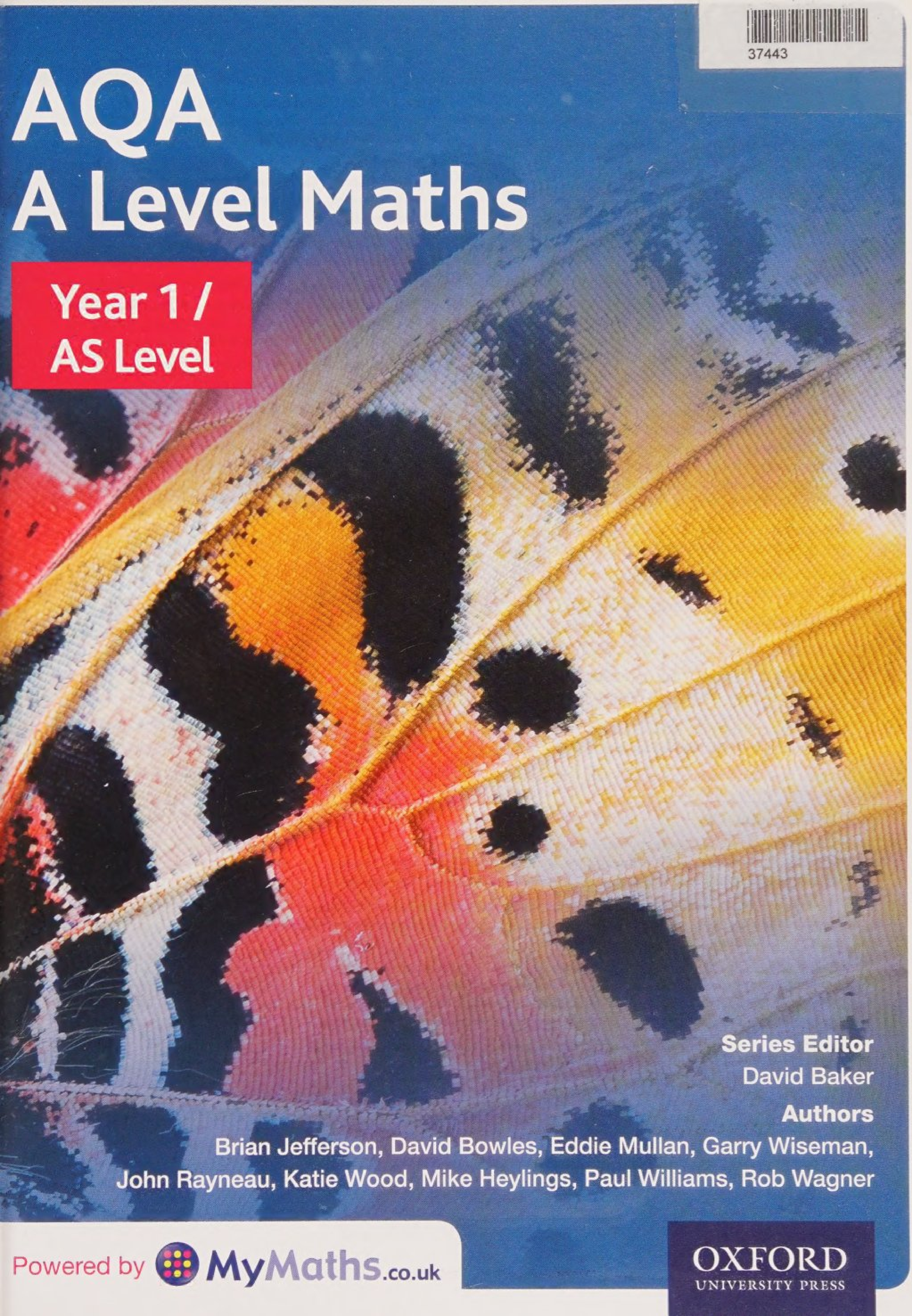

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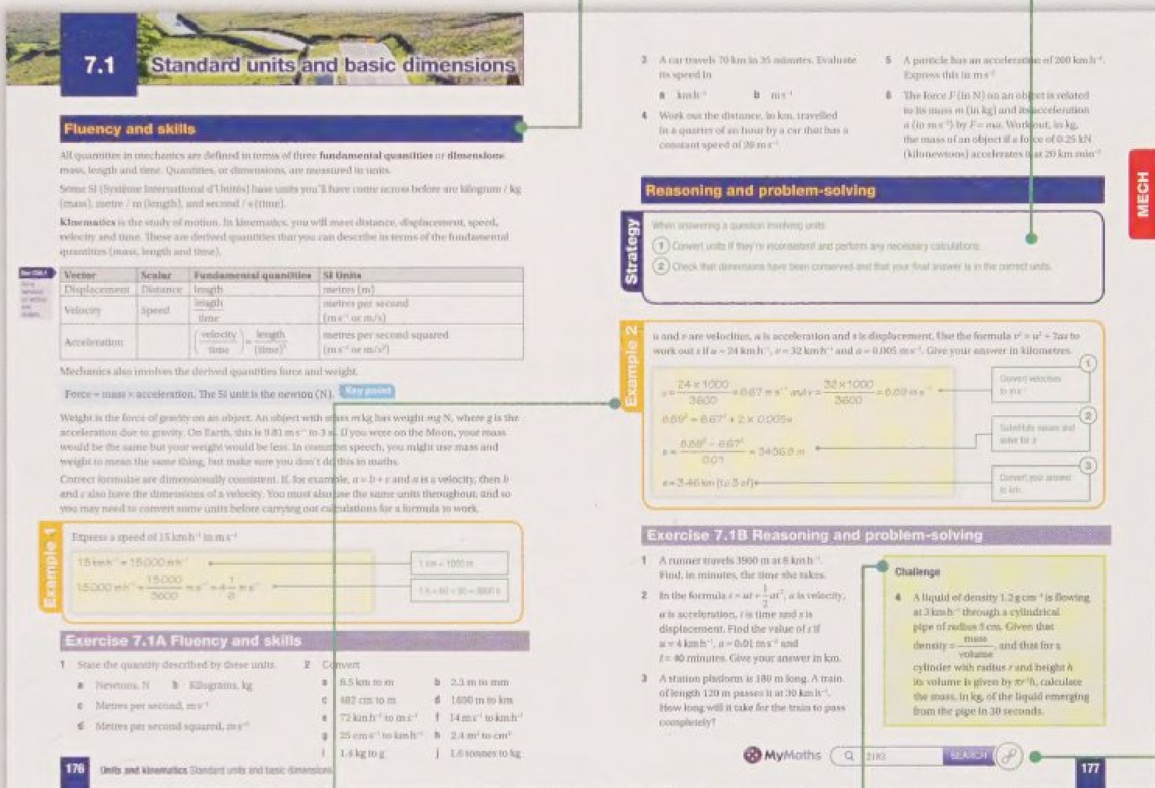
Contents

About this book

This book has been specifically created for those studying the AQA 2017 Mathematics AS and A Levels. It's been written by a team of experienced authors and teachers, and it's packed with questions, explanation and extra features to help you get the most out of your course.

Every section starts by covering the basic **Fluency and skills** (A01), then builds on these techniques by looking at **Reasoning and problem-solving** (A02 and A03).

Strategy boxes help build problem-solving techniques.

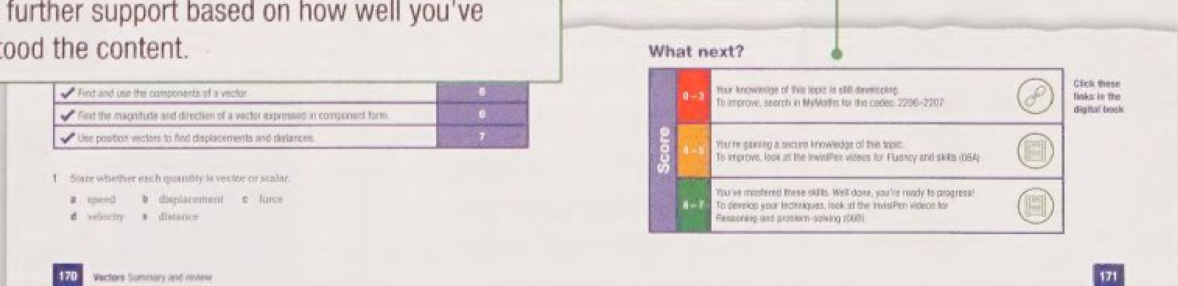


Worked examples provide a sample answer and commentary to practice questions. The circled numbers show how each step is linked to the strategy box.

Challenge questions in each section stretch you with questions at the highest level of demand. **Answers to all questions** are in the back of this book, and **full solutions are available free** online.

Links to **MyMaths** provide a quick route to **extra support and practice**. Just log in and key the code into the search bar.

At the end of each chapter, a **What Next** box provides links to further support based on how well you've understood the content.



Support for when and how to use **calculators** is available throughout this book, with links to further demonstrations in the **digital book**.

Try it on your calculator

You can use a calculator to evaluate the gradient of the tangent to a curve at a given point.

d/dx(5X² - 2X, 3)

28

Activity

Find out how to calculate the gradient of the tangent to the curve $y = 5x^2 - 2x$ where $x = 3$ on *your* calculator.

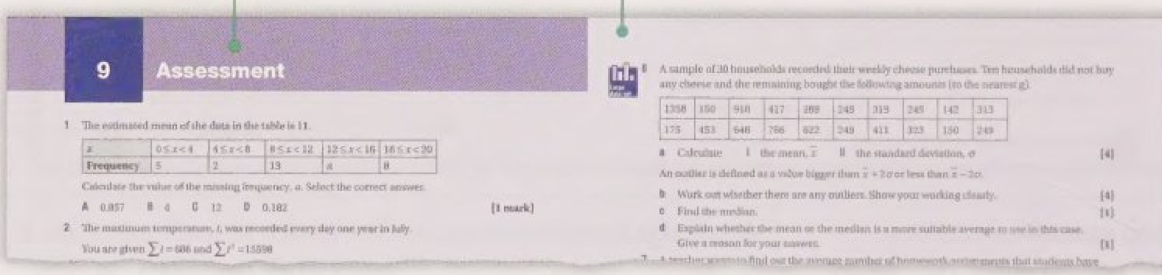
ICT Resource online

To investigate gradients of chords for a graph, click this link in the digital book.

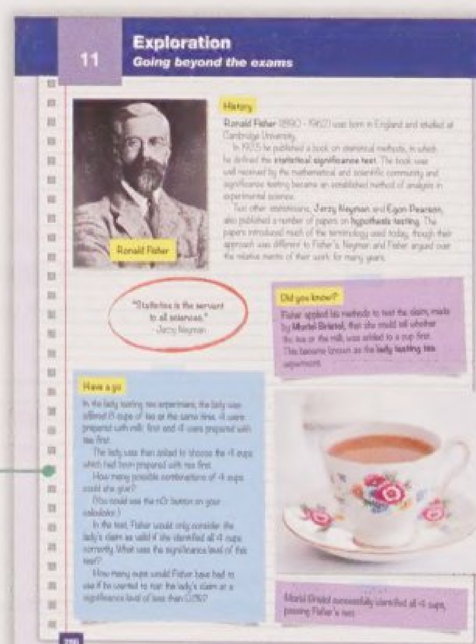
Links to **ICT resources** on Kerboodle show how technology can be used to help understand the maths involved.

Assessment sections at the end of each chapter test everything covered within that chapter. Further **synoptic assessments** covering Pure, Mechanics and Statistics can be found at the end of chapters 5, 8 and 11

Dedicated questions throughout the statistics chapters will familiarise you with the **large data set.**



On the chapter **Introduction page**, the **Orientation box** explains what you should already know, what you will learn in this chapter, and what this leads to.



At the end of every chapter, an **Exploration page** gives you an opportunity to explore the subject beyond the specification.

1 Algebra 1

Supply and demand is a well-known example of how maths helps us model real situations that occur in the world. Economists and business analysts use simultaneous equations to model how changes in price will affect both the supply of, and demand for, a particular product. This allows them to forecast the optimum price–the price at which both supply and demand are optimised.

Algebra is a branch of maths that includes simultaneous equations, along with many other topics such as inequalities, surds and polynomial functions. Algebra is used to model real world occurrences in fields such as economics, engineering and the sciences, and so an understanding of algebra is important in a wide range of different situations.

Orientation

What you need to know

KS4

- Understand and use algebraic notation and vocabulary.
- Simplify and manipulate algebraic expressions.
- Rearrange formulae to change the subject.
- Solve linear equations.

What you will learn

- To use direct proof, proof by exhaustion and counter examples.
- To use and manipulate index laws.
- To manipulate surds and rationalise a denominator.
- To solve quadratic equations and sketch quadratic curves.
- To understand and use coordinate geometry.
- To understand and solve simultaneous equations.
- To understand and solve inequalities.

What this leads to

Ch12 Algebra 2
Functions.
Parametric equations.
Algebraic and partial fractions.

Ch17 Numerical Methods
Simple and iterative root finding.
Newton-Raphson root finding.

MyMaths Practise before you start: 1170, 1171, 1928, 1929

1.1 Argument and proof

Fluency and skills

A **proof** is a logical argument for a mathematical statement. It shows that something *must* be either true or false.

The most simple method of proving something is called **direct proof**. It's sometimes also called deductive proof. In direct proof, you rely on statements that are already established, or statements that can be assumed to be true, to show by deduction that another statement is true (or untrue).

> Statements that can be assumed to be true are sometimes known as **axioms**.

Examples of statements that can be assumed to be true include 'you can draw a straight line segment joining any two points,' and 'you can write all even numbers in the form $2n$ and all odd numbers in the form $2n-1$'.

Key point

To use direct proof you

- Assume that a statement, P, is true.
- Use P to show that another statement, Q, must be true.

Example 1

Use direct proof to prove that the square of any integer is one more than the product of the two integers either side of it.

Let the integer be n

The two numbers on each side of n are $n-1$ and $n+1$ — *Assume that this statement is true.*

The product of these two numbers is $(n-1)(n+1)$

$(n-1)(n+1)=n^2-1$ — *Expand the brackets to get the square number n^2*

So $n^2=(n-1)(n+1)+1$ — *Rearrange to show the required result.*

So the square of any integer is one more than the product of the two integers either side of it.

Another method of proof is called **proof by exhaustion**. In this method, you list all the possible cases and test each one to see if the result you want to prove is true. All cases must be true for proof by exhaustion to work, since a single counter example would disprove the result.

Key point

To use proof by exhaustion you

- List a set of cases that exhausts all possibilities.
- Show the statement is true in each and every case.

Example 2

Prove, by exhaustion, that $p^2 + 1$ is not divisible by 3, where p is an integer and $6 \le p \le 10$

p	$p^2 + 1$	Divisible by 3?
6	37	NO
7	50	NO
8	65	NO
9	82	NO
10	101	NO

In all cases, $p^2 + 1$ is not divisible by 3

Write down every possible value of p and calculate $p^2 + 1$

Show that the statement is true in each case.

You can also *disprove* a statement with **disproof by counter example**, in which you need to find just one example that does not fit the statement.

Example 3

Prove, by counter example, that the statement '$n^2 + n + 1$ is prime for all integers n' is false.

Let $n = 4$

$4^2 + 4 + 1 = 21 = 3 \times 7$

21 has factors 1, 3, 7 and 21, so is not prime.

This disproves the statement '$n^2 + n + 1$ is prime for all integers n'

Show that the statement is false for one value of n

Exercise 1.1A Fluency and skills

Use direct proof to answer these questions.

1 Prove that the number 1 is *not* a prime number.

2 Prove that the sum of two odd numbers is always even.

3 Prove that the product of two consecutive odd numbers is one less than a multiple of four.

4 Prove that the mean of three consecutive integers is equal to the middle number.

5 a Prove that the sum of the squares of two consecutive integers is odd.

b Prove that the sum of the squares of two consecutive even numbers is always a multiple of four.

6 Show that the sum of four consecutive positive integers has both even factors and odd factors greater than one.

7 Prove that the square of the sum of any two positive numbers is greater than the sum of the squares of the numbers.

8 Prove that the perimeter of an isosceles right-angled triangle is always greater than three times the length of one of the equal sides.

9 a and b are two numbers such that $a = b - 2$ and the sum and product of a and b are equal. Prove that neither a nor b is an integer.

10 If $(5y)^2$ is even for an integer y, prove that y must be even.

Use proof by exhaustion in the following questions.

11 Prove that there is exactly one square number and exactly one cube number between 20 and 30

12 Prove that, if a month has more than five letters in its name, a four letter word can be made using those letters.

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13 Prove that, for an integer x, $(x+1)^3 \geq 3^x$ for $0 \leq x \leq 4$

14 Prove that no square numbers can have a last digit 2, 3, 7 or 8

Give counter examples to disprove these statements.

15 The product of two prime numbers is always odd.

16 When you throw two six-sided dice, the total score shown is always greater than six.

17 When you subtract one number from another, the answer is always less than the first number.

18 Five times any number is always greater than that number.

19 If $a > b$, then $a^b > b^a$

20 The product of three consecutive integers is always divisible by four.

Reasoning and problem-solving

Strategy

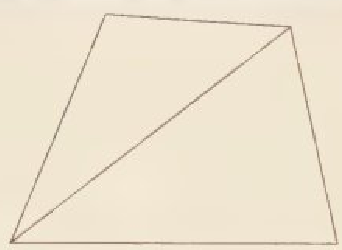
To prove or disprove a statement

1. Decide which method of proof to use.
2. Follow the steps of your chosen method.
3. Write a clear conclusion that proves/disproves the statement.

Example 4

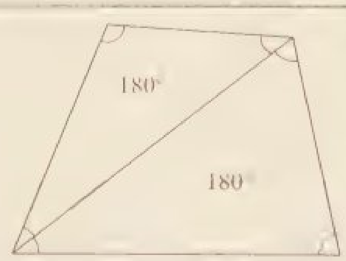
Prove that the sum of the interior angles in any convex quadrilateral is 360°

The sum of the interior angles of any quadrilateral can be found by breaking the quadrilateral into two triangles.



(1) Since you know that the interior angles of a triangle sum to 180°, you use this to try to prove the result by direct proof.

The sum of the interior angles of any triangle equals 180°, and each of the two triangles will contribute 180° to the total sum of all angles in the quadrilateral.



(2) Apply the result about angles in a triangle to angles in a quadrilateral.

So the interior angle sum of a convex quadrilateral is the same as the sum of the interior angles of two triangles, which is 360°

(3) Write your conclusion clearly.

Example 5

Jane says that there are exactly three prime numbers between the numbers 15 and 21 (inclusive).

Is she correct? Use a suitable method of proof to justify your answer.

NUMBER	PRIME?
15	NO → $15 = 1 \times 15, 3 \times 5$
16	NO → $16 = 1 \times 16, 2 \times 8, 4 \times 4$
17	YES → $17 = 1 \times 17$
18	NO → $18 = 1 \times 18, 2 \times 9, 3 \times 6$
19	YES → $19 = 1 \times 19$
20	NO → $20 = 1 \times 20, 2 \times 10, 4 \times 5$
21	NO → $21 = 1 \times 21, 3 \times 7$

There are exactly two prime numbers between 15 and 21, so Jane is wrong (she said there were three).

(1)(2) Use proof by exhaustion to check all the numbers within the range of values.

(3) Write a clear conclusion that proves or disproves the statement.

Exercise 1.1B Reasoning and problem-solving

1 P is a prime number and Q is an odd number.

a Sue says PQ is even, Liz says that PQ is odd and Graham says PQ could be either. Who is right? Use a suitable method of proof to justify your answer.

b Sue now says that $P(Q+1)$ is always even. Is she correct? Use a suitable method of proof to justify your answer.

2 Use a suitable method of proof to show whether the statement '*Any odd number between 90 and 100 is either a prime number or the product of only two prime numbers*' is true or false.

3 Use a suitable method of proof to prove that the value of $9^n - 1$ is divisible by 8 for $1 \le n \le 6$

4 Is it true that 'all triangles are obtuse'? Use a suitable method of proof to justify your answer.

5 Prove that the sum of the interior angles of a convex hexagon is 720°

6 Martin says that 'All quadrilaterals with equal sides are squares.' Use a suitable method of proof to show if his statement is true or false.

7 Prove that the sum of the interior angles of a convex n-sided polygon is $180(n-2)°$

8 Use a suitable method of proof to prove or disprove the statement 'If $m^2 = n^2$ then $m = n$.'

9 The hypotenuse of a right-angled triangle is $(2s+a)$ cm and one other side is $(2s-a)$ cm. Use a suitable method of proof to show that the square of the remaining side is a multiple of eight.

10 Use a suitable method of proof to show that, for $1 \le n \le 5$,

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{n\times(n+1)}=\frac{n}{n+1}$$

Challenge

11 A teacher tells her class that any number is divisible by three if the sum of its digits is divisible by three. Use a suitable method to prove this result for two-digit numbers.

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1.2 Index laws

Fluency and skills

The algebraic term $3x^5$ is written in **index form**. The 3 is called the **coefficient**. The x part of the term is called the **base**. The 5 is called the **power**, or **index**, or **exponent**. $3x^5$ means $3 \times x \times x \times x \times x \times x$

Indices follow some general rules.

Key point

Rule 1: Any number raised to the power zero is 1 $\quad x^0 = 1$

Rule 2: Negative powers may be written as reciprocals. $\quad x^{-n} = \dfrac{1}{x^n}$

Rule 3: Any base raised to the power of a unit fraction is a root. $\quad x^{\frac{1}{n}} = \sqrt[n]{x}$

Rule 1 has an exception when $x = 0$, as 0^0 is undefined.

$x^{\frac{1}{2}} = \sqrt{x}$ and $x^{\frac{1}{3}} = \sqrt[3]{x}$

You don't normally write the '2' in a square root.

You can combine terms in index form by following this simple set of rules called the **index laws**.

To use the index laws, the bases of all the terms must be the same.

Key point

Law 1: To multiply terms you add the indices. $\quad x^a \times x^b = x^{a+b}$

Law 2: To divide terms you subtract the indices. $\quad x^a \div x^b = x^{a-b}$

Law 3: To raise one term to another power you multiply the indices. $\quad (x^a)^b = x^{a \times b}$

By combining the third general rule and the third index law you can see that $\sqrt[b]{(x^a)} = x^{\frac{a}{b}} = (\sqrt[b]{x})^a$

So $\sqrt[3]{(125^4)} = (\sqrt[3]{125})^4 = 5^4 = 625$

Example 1

Simplify these expressions, leaving your answers in index form.

a $2m^4n^2 \times 3m^3n^9$ $\qquad$ **b** $4d^{\frac{5}{3}} \div 2d^{\frac{1}{3}}$

a $2 \times 3 \times m^{4+3} \times n^{2+9} = 6m^7n^{11}$ — Use $x^a \times x^b = x^{a+b}$

b $4d^{\frac{5}{3}-\frac{1}{3}} \div 2 = 2d^{\frac{4}{3}}$ — Use $x^a \div x^b = x^{a-b}$

Exercise 1.2A Fluency and skills

Simplify the expressions in questions **1** to **44**. Show your working.

1 4^3

2 $(-3)^5$

3 $7^8 \div 7^1$

4 $c^7 \times c^4$

5 $(-p^3)^4$

6 $-(p^3)^4$

7 $(2c^{-1})^6$

8 $d^7 \times d^3 \times d^4$

9 $2e^3 \times 5e^4 \times 7e^2$

10 $4f^2 \times -3f^4 \times 9f^6$

11 $24g^{12} \div 6g^6$

12 $-44k^{44} \div 11k^{-11}$

13 $12f^2 \times 4f^4 \div 6f^3$

14 $12e^{13} \div 6e^4 \div 3e^7$

15 $3a \times 5b$

16 $5w \times 4x \times (-6x)$

17 $2d \times 3e \times 4f^2$

18 $3h^6 \times (-3h^8)$

19 $5r^5s^6 \times r^3s^4$

20 $5r^5s^6 \div r^3s^4$

21 $(g^2h^3) \times (-g^7h^5) \times (ghi^4)$

22 $(g^2h^3) \times (-g^7h^5) \div (ghi^4)$

23 $(-20z^9y^6) \div (-4z^4y)$

24 $\sqrt{36u^{36}}$

25 $(36u^{36})^{\frac{1}{2}}$

26 $\sqrt[3]{125t^{27}}$

27 $\sqrt[3]{-125t^{27}c^{12}}$

28 $(5)^{-1}$

29 $\left(\frac{1}{5}\right)^{-1}$

30 $6u^0$

31 $-50f^0$

32 $7y^0 - 4z^0$

33 4^{-2}

34 2^{-10}

35 $(3w)^{-2}$

36 $(3w^{-2})^{-2}$

37 $64^{\frac{3}{2}}$

38 $1024^{\frac{1}{5}}$

39 $1024^{\frac{4}{5}}$

40 $16^{\frac{3}{4}}$

41 $16^{\frac{-3}{4}}$

42 $\left(\frac{36}{49}\right)^{\frac{1}{2}}$

43 $\left(\frac{36}{49}\right)^{\frac{-1}{2}}$

44 $\left(\frac{36}{49}\right)^{\frac{3}{2}}$

45 If $5^n = 625$, find the value of n

46 If $3^m = 243$, find the value of m

47 If $6^{2t+1} = 216$, find the value of t

48 If $(2^{2b})(2^{-6b}) = 256$, find the value of b

Reasoning and problem-solving

Strategy

To solve problems involving indices

1. Use the information in the question to write an expression or equation involving indices.
2. Apply the laws of indices correctly.
3. Simplify expressions as much as possible.
4. Give your answer in an appropriate format that is relevant to the question.

Example 2

A swimming pool has a volume of $16s^2$ cubic metres.

a How long does it take to fill, from empty, if water is pumped in at a rate of $4s^{-3}$ cubic metres per minute?

b If it takes 128 minutes to fill the swimming pool, calculate the value of s

a Let the time taken to fill the swimming pool be t minutes, the volume of the swimming pool be V and the rate of water flow be r

$t = \frac{V}{r} = \frac{16s^2}{4s^{-3}}$ — 1 Write an equation involving indices.

$= 16s^2 \div 4s^{-3}$

$= 4s^{2-(-3)}$ — 2 Apply the laws of indices.

$= 4s^5$ — 3 Simplify.

So it takes $4s^5$ minutes to fill the swimming pool. — 4 Give your answer in an appropriate format.

b $t = 4s^5$

so $128 = 4s^5$

$s^5 = 32$

$s = 2$

Example 3

A rectangular flower bed has sides of length x and $8x$

Around it are 6 further flower beds, each with an area equal to the cube root of the larger flower bed.

Calculate the total area covered by the 6 smaller flower beds, giving your answer in index form.

Area of large flower bed $= x \times 8x$

Total area of smaller flower beds

$= 6 \times \sqrt[3]{x \times 8x}$ — (1) Write an equation for the total area.

$= 6 \times \sqrt[3]{8x^2}$

$= 6 \times 8^{\frac{1}{3}} \times x^{\frac{2}{3}}$ — (2) Apply the laws of indices.

$= 12x^{\frac{2}{3}}$ — (3)(4) Simplify and give the final answer in index form.

Example 4

The brain mass (kg) of a species of animal is approximately one hundredth of the cube root of the square of its total body mass (kg).

a Write a formula relating brain mass, B, to total body mass, m, using index form.

b Use your formula to calculate

  i The approximate brain mass of an animal of mass 3.375 kg,

  ii The approximate total body mass of an animal with brain mass 202.5 g.

a $B = \dfrac{\sqrt[3]{m^2}}{100}$ — (1) Write a formula for the brain mass.

$= \dfrac{m^{\frac{2}{3}}}{100}$ — (2) Apply the index laws correctly.

b i $B = \dfrac{3.375^{\frac{2}{3}}}{100}$

$= 0.0225$ kg or 22.5 g — (4) Give your answer in an appropriate form.

ii $0.2025 = \dfrac{m^{\frac{2}{3}}}{100}$ — Convert the brain mass into kilograms and substitute into the formula.

$m^{\frac{2}{3}} = 20.25$

$m = 20.25^{\frac{3}{2}}$ — Use inverse operations to find m

$= 91.125$ kg

Exercise 1.2B Reasoning and problem-solving

1 a Work out the area of a square of side $2s^2$ inches.

b Work out the side length of a square of area $25p^4q^6$ cm^2.

2 a Work out the circumference and area of a circle of radius $3w^5$ ft.

b The volume of a sphere is $\frac{4}{3}\pi\times\text{radius}^3$ and the surface area is $4\pi\times\text{radius}^2$.

Work out the surface area and volume of a sphere of radius $3w^4$ ft.

3 What term multiplies with $4c^2d^3$, $5de^2$ and $3c^2e^3$ to give 360?

4 Work out the volume of a cuboid with dimensions $4p^2q^3$, $3pq^2$ and $\sqrt{9p^4q^0}$
Give your answer in index form.

5 a The area of a rectangle is $8y^5z^7$ and its length is $4y^2z^3$. Work out its width.

b Explain why the area of a rectangle of sides $\sqrt[3]{8m^{-3}n^6}$ and $\sqrt{16m^2n^{-4}}$ is independent of m and n. What is the area?

6 A cyclist travels $4b^2c^{\frac{1}{2}}$ miles in $3b^2c$ hours. What is her average speed?

7 a The volume of a cylinder is $8\pi c^2d$ cm^3. The radius of the cylinder is $2cd^{-1}$ cm. What is its height?

b Explain why the volume of a cylinder of radius $3s^2t^{-1}$ and height $(5st)^2$ is independent of t. What is the volume?

8 A disc of radius $3v^2z^{-2}$ cm is removed from a disc of radius $4v^2z^{-2}$ cm. What is the remaining area?

9 a Work out the hypotenuse of a right-angled triangle with perpendicular sides of length $5n^{\frac{1}{2}}$ and $12n^{\frac{1}{2}}$

b Work out the area of the right-angled triangle described in part **a**

10 a To work out the voltage, V volts, in a circuit with current i amps and resistance r ohms, you multiply the current and resistance together.

Work out the voltage in a circuit of resistance $3m^4n^{-4}$ ohms carrying a current of $6m^{-2}n^{-3}$ amps.

b The power, W watts, in a circuit with current i amps and resistance r ohms, is found by multiplying the resistance by the square of the current. Work out the power when the current and resistance are the same as in the circuit in part **a**

11 The kinetic energy of a body is given by $\frac{1}{2}\times\text{mass}\times\text{velocity}^2$
Work out the kinetic energy when mass $= m$ and velocity $= 9x^{\frac{3}{4}}c^{\frac{3}{4}}$

12 You can find your Body Mass Index (BMI) by dividing your mass (kg) by the square of your height (m). If your mass is $3gt$ kg and your height is $4gt^{-1}$ m, what is your BMI?

13 In an electrical circuit, the total resistance of two resistors, t_1 and t_2, connected in parallel, is found by dividing the product of their resistances by the sum of their resistances. Work out the total resistance when $t_1 = 3rs^2$ ohms and $t_2 = 5rs^2$ ohms.

Challenge

14 In a triangle with sides of length a, b and c the semi-perimeter, s, is half the sum of the three sides. The area of the triangle can be found by subtracting each of the sides from the semi-perimeter in turn (to give three values), multiplying these expressions and the semi-perimeter altogether and then square rooting the answer.

a Write a formula for the area of the triangle involving s, a, b and c

b Use your formula to work out the area of a triangle with sides, $12xy$, $5xy$ and $13xy$

c You could have found the area of this triangle in a much easier way. Explain why.

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1.3 Surds

Fluency and skills

Key point

A **rational number** is one that you can write exactly in the form

$$\frac{p}{q}$$

where p and q are integers, $q \neq 0$

Numbers that you cannot write exactly in this form are **irrational numbers**. If you express them as decimals, they have an infinite number of non-repeating decimal places.

Some roots of numbers are irrational, for example, $\sqrt{3} = 1.732...$ and $\sqrt[3]{10} = 2.15443...$ are irrational numbers.

Key point

Irrational numbers involving roots, $\sqrt[n]{\ }$ or $\sqrt{\ }$, are called **surds**.

You can use the following laws to simplify surds

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

You usually write surds in their simplest form, with the smallest possible number written inside the root sign.

You can simplify surds by looking at their factors.

You should look for factors that are square numbers.

For example $\sqrt{80} = \sqrt{16} \times \sqrt{5} = 4\sqrt{5}$

If $\sqrt{a}$ and $\sqrt{b}$ cannot be simplified, then you cannot simplify $\sqrt{a} + \sqrt{b}$ or $\sqrt{a} - \sqrt{b}$ for $a \neq b$

Example 1

Simplify these expressions. Show your working.

a $\sqrt{7} \times \sqrt{7}$ **b** $\sqrt{5} \times \sqrt{20}$ **c** $\sqrt{\frac{1}{9}} \times \sqrt{9}$

d $\sqrt{80} - \sqrt{20}$ **e** $\sqrt{63} + \sqrt{112}$ **f** $\sqrt{\frac{4}{3}} + \sqrt{\frac{25}{3}}$

a $\sqrt{7} \times \sqrt{7} = 7$ **b** $\sqrt{5} \times \sqrt{20} = \sqrt{100} = 10$ **c** $\sqrt{\frac{1}{9}} \times \sqrt{9} = \frac{1}{3} \times 3 = 1$

d $4\sqrt{5} - 2\sqrt{5} = 2\sqrt{5}$ **e** $3\sqrt{7} + 4\sqrt{7} = 7\sqrt{7}$ **f** $2\sqrt{\frac{1}{3}} + 5\sqrt{\frac{1}{3}} = 7\sqrt{\frac{1}{3}}$

Calculations are often more difficult if surds appear in the denominator. You can simplify such expressions by removing any surds from the denominator. To do this, you multiply the numerator and denominator by the same value to find an **equivalent fraction** with surds in the numerator only. This is easier to simplify.

This process is called **rationalising the denominator**.

Key point

If the fraction is in the form

$\frac{k}{\sqrt{a}}$, multiply numerator and denominator by $\sqrt{a}$

$\frac{k}{a\pm\sqrt{b}}$, multiply numerator and denominator by $a\mp\sqrt{b}$

$\frac{k}{\sqrt{a}\pm\sqrt{b}}$, multiply numerator and denominator by $\sqrt{a}\mp\sqrt{b}$

Example 2

Rationalise these expressions. Show your working.

a $\frac{4}{\sqrt{5}}$ **b** $\frac{6+\sqrt{7}}{9-\sqrt{7}}$

a $\frac{4}{\sqrt{5}}=\frac{4}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\frac{4\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}=\frac{4\sqrt{5}}{5}$ — Multiply top and bottom by $\sqrt{5}$

b $\frac{6+\sqrt{7}}{9-\sqrt{7}}=\frac{6+\sqrt{7}}{9-\sqrt{7}}\times\frac{9+\sqrt{7}}{9+\sqrt{7}}$ — Multiply top and bottom by $9+\sqrt{7}$

$$=\frac{54+9\sqrt{7}+6\sqrt{7}+7}{(9-\sqrt{7})(9+\sqrt{7})}$$

$$=\frac{61+15\sqrt{7}}{81-9\sqrt{7}+9\sqrt{7}-\sqrt{7}\sqrt{7}}$$

$$=\frac{61+15\sqrt{7}}{81-9\sqrt{7}+9\sqrt{7}-7}=\frac{61+15\sqrt{7}}{74}$$

Exercise 1.3A Fluency and skills

Complete this exercise without a calculator.

1 Classify these numbers as rational or irrational.

a $1+\sqrt{25}$ **b** π^2

c $4-\sqrt{3}$ **d** $\sqrt{21}$

e $\sqrt{169}$ **f** $\left(\sqrt{8}\right)^2$

g $\left(\sqrt{17}\right)^3$

2 For each of these expressions, show that they can be written in the form $a\sqrt{b}$ where a and b are integers.

a $\sqrt{4}\times\sqrt{21}$ **b** $\sqrt{8}\times\sqrt{7}$

c $\sqrt{75}$ **d** $\sqrt{27}$

e $\frac{\sqrt{800}}{10}$ **f** $\left(\sqrt{8}\right)^3$

g $\left(\sqrt{17}\right)^3$ **h** $2\sqrt{3}\times3\sqrt{2}$

i $5\sqrt{6}\times7\sqrt{18}$ **j** $4\sqrt{24}\times6\sqrt{30}$

3 Show that these expressions can be expressed as positive integers.

a $\frac{\sqrt{128}}{\sqrt{2}}$ **b** $\frac{\sqrt{125}}{\sqrt{5}}$

4 Show that these expressions can be written in the form $\frac{a}{b}$, where a and b are positive integers.

a $\frac{\sqrt{32}}{\sqrt{200}}$ **b** $\frac{\sqrt{50}}{\sqrt{72}}$

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5 Show that these expressions can be written in the form $a\sqrt{b}$, where a and b are integers.

a $\sqrt{54}$ b $\sqrt{432}$

c $\sqrt{1280}$ d $\sqrt{3388}$

e $\sqrt{2}\times\sqrt{20}$ f $\sqrt{2}\times\sqrt{126}$

g $\sqrt{20}+\sqrt{5}$ h $\sqrt{18}-\sqrt{2}$

i $\sqrt{150}-\sqrt{24}$ j $\sqrt{75}+\sqrt{12}$

k $\sqrt{27}-\sqrt{3}$ l $\sqrt{5}+\sqrt{45}$

m $\sqrt{363}-\sqrt{48}$ n $\sqrt{72}-\sqrt{288}+\sqrt{200}$

6 Show these expressions can be written in the form $a+b\sqrt{c}$, where a, b and c are integers.

a $\left(3\sqrt{6}+\sqrt{5}\right)^2$

b $\left(\sqrt{2}+3\right)\left(4+\sqrt{2}\right)$

c $\left(\sqrt{2}-3\right)\left(4-\sqrt{2}\right)$

d $\left(3\sqrt{5}+4\right)\left(2\sqrt{5}-6\right)$

e $\left(5\sqrt{3}+3\sqrt{2}\right)\left(4\sqrt{27}-5\sqrt{8}\right)$

7 Rationalise the denominators in these expressions and leave your answers in their simplest form. Show your working.

a $\dfrac{1}{\sqrt{13}}$ b $\dfrac{8}{\sqrt{6}}$

c $\dfrac{\sqrt{11}}{2\sqrt{5}}$ d $\dfrac{3}{\sqrt{2}-1}$

e $\dfrac{3\sqrt{7}\times5\sqrt{4}}{6\sqrt{7}}$ f $\dfrac{13\sqrt{15}-2\sqrt{10}}{4\sqrt{75}}$

g $\dfrac{5}{8-\sqrt{5}}$ h $\dfrac{2\sqrt{2}}{4+\sqrt{2}}$

i $\dfrac{\sqrt{6}-\sqrt{5}}{\sqrt{6}+\sqrt{5}}$ j $\dfrac{3\sqrt{11}-4\sqrt{7}}{\sqrt{11}-\sqrt{7}}$

8 Rationalise the denominators and simplify these expressions. a, b and c are integers.

a $\dfrac{a+\sqrt{b}}{\sqrt{b}}$ b $\dfrac{a+\sqrt{b}}{a-\sqrt{b}}$

c $\dfrac{\sqrt{a}+b\sqrt{c}}{b\sqrt{c}}$ d $\dfrac{\sqrt{a}-b\sqrt{c}}{\sqrt{a}+\sqrt{b}}$

Reasoning and problem-solving

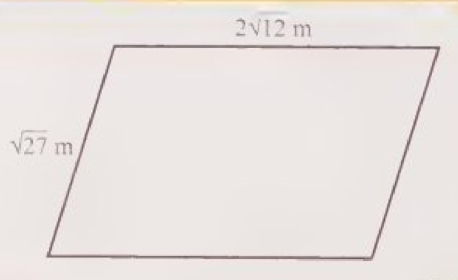
Strategy

To solve problems involving surds

1. Use the information given to form an expression involving surds.
2. If possible, simplify surds. They should have the lowest possible number under the root sign.
3. Rationalise any denominator containing surds.

Example 3

The sides of a parallelogram are $\sqrt{27}$ m and $2\sqrt{12}$ m, and it has a perpendicular height of $\dfrac{10}{\sqrt{3}}$ m.



a Work out the perimeter of the parallelogram.

b Work out the area of the parallelogram.

Give your answers in their simplest form.

a Perimeter

$2(\sqrt{27}+2\sqrt{12})=2(3\sqrt{3}+4\sqrt{3})=14\sqrt{3}$ m

b Area = base × perpendicular height

$\sqrt{27}\times\dfrac{10}{\sqrt{3}}=\dfrac{3\sqrt{3}\times10}{\sqrt{3}}=30\text{ m}^2$

(1) Form an expression involving surds.

(2) Simplify

(2)(3) Simplify and rationalise the denominator.

Exercise 1.3B Reasoning and problem-solving

1 A rectangle has sides of length $2\sqrt{3}$ cm and $3\sqrt{2}$ cm. What is its area? Show your working.

2 a A circle has radius $9\sqrt{3}$ cm. Show that its area is $243\pi\,\text{cm}^2$.

b A circle has area $245\pi\,\text{m}^2$. What is its diameter? Show your working.

3 a A car travels $18\sqrt{35}$ m in $6\sqrt{7}$ s. Work out its speed, showing your working.

b A runner travels for 5 s at $\frac{8}{\sqrt{5}}\,\text{m s}^{-1}$.

Work out how far she ran in simplified form. Show your working.

4 A cube has sides of length $(2+\sqrt{7})$ m. Work out its volume in simplified surd form.

5 Rectangle A has sides of length $3\sqrt{3}$ m and $\sqrt{5}$ m. Rectangle B has sides of length $\sqrt{5}$ m and $\sqrt{7}$ m. How many times larger is rectangle A than rectangle B? Give your answer in its simplest surd form, showing your working.

6 A right-angled triangle has perpendicular sides of $2\sqrt{3}$ cm and $3\sqrt{7}$ cm. Calculate the length of the hypotenuse. Show your working and give your answer in simplified form.

7 Base camp is $5\sqrt{5}$ miles due east and $5\sqrt{7}$ miles due north of a walker. What is the exact distance from the walker to the camp? Show your working.

8 The arc of a bridge forms part of a larger circle with radius $\frac{12}{\sqrt{3}}$ m. If the arc of the bridge subtends an angle of 45°, show that the length of the bridge is $\sqrt{3}\,\pi$ m.

9 The equation of a parabola is $y^2 = 4ax$

Find y when $a = 6 - \sqrt{6}$ and $x = \frac{6+\sqrt{6}}{10}$

Show your working and give your answer in simplified form.

10 Show that the ratio of the volumes of two cubes of sides $6\sqrt{8}$ cm and $4\sqrt{2}$ cm is 27

11 The top speeds, in m s^{-1}, of two scooters are given as $\frac{12}{\sqrt{a}}$ and $\frac{17\sqrt{a}}{3a}$, where a is the volume of petrol in the tank. Find the difference in top speed between the two scooters if they both contain the same volume of petrol. Give your answer in surd form, showing your working.

12 The area of an ellipse with semi-diameters a and b is given by the formula πab

Work out the area of an ellipse where

$a = \frac{5}{4+\sqrt{3}}$ m and $b = \frac{8}{4-\sqrt{3}}$ m

Show your working.

13 The force required to accelerate a particle can be calculated using $F = ma$, where F is the force, m is the mass of the particle and a is the acceleration. Showing your working, find F when $m = 8\sqrt{6}$ and $a = \frac{5}{2+\sqrt{6}}$

14 An equilateral triangle with side length $5\sqrt{6}$ inches has one vertex at the origin and one side along the positive x-axis.

The centre is on the vertical line of symmetry, $\frac{1}{3}$ of the way from the x-axis to the vertex.

Work out the distance from the origin to the centre of mass of the triangle. Show your working.

Challenge

15 The Indian Mathematician Brahmagupta (598 – 670) developed a formula to calculate the area of a cyclic quadrilateral.

If the sides of the quadrilateral are a, b, c and d, and $s = \frac{a+b+c+d}{4}$, the area is

$A = \sqrt{s(s-a)(s-b)(s-c)(s-d)}$

A quadrilateral with side lengths $(1+\sqrt{2})$, $(3+2\sqrt{2})$, $(5+3\sqrt{2})$ and $(7+4\sqrt{2})$ cm is inscribed in a circle.

Prove that $A = \left(\frac{9}{2}+3\sqrt{2}\right)\sqrt{4+\frac{5\sqrt{2}}{2}}$

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1.4 Quadratic functions

Fluency and skills

A **quadratic function** can be written in the form $ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$

A **quadratic equation** can be written in the general form $ax^2 + bx + c = 0$

Curves of quadratic functions, $y = ax^2 + bx + c$, have the same general shape. The curve crosses the y-axis when $x = 0$, and the curve crosses the x-axis at any **roots** (or solutions) of the equation $ax^2 + bx + c = 0$

Quadratic curves are symmetrical about their **vertex** (the turning point). For $a > 0$, this vertex is always a **minimum** point, and for $a < 0$ this vertex is always a **maximum** point.

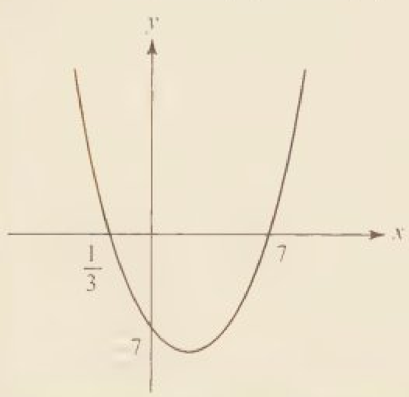
When $a > 0$, a quadratic graph looks like this.

When $a < 0$, a quadratic graph looks like this.

Example 1

The quadratic equation $3x^2 - 20x - 7 = 0$ has solutions $x = -\frac{1}{3}$ and $x = 7$
Sketch the curve of $f(x) = 3x^2 - 20x - 7$, showing where it crosses the axes.

When $x = 0$, $y = 3(0)^2 - 20(0) - 7 = -7$ — Find the y-intercept

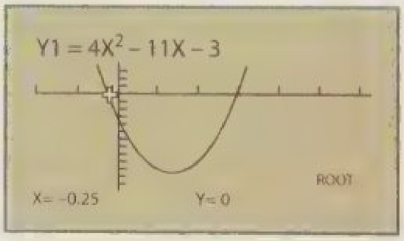


The curve $y = f(x)$ crosses the x-axis at the solutions to $f(x) = 0$

Calculator

Try it on your calculator

You can sketch a curve on a graphics calculator.



Activity
Find out how to sketch the curve $y = 4x^2 - 11x - 3$ on *your* graphics calculator.

You can solve some quadratics in the form $ax^2 + bx + c = 0$ by **factorisation**. To factorise a quadratic, try to write it in the form $(mx + p)(nx + q) = 0$

Key point

A quadratic equation that can be written in the form $(mx + p)(nx + q) = 0$ has solutions $x = -\frac{p}{m}$ or $x = -\frac{q}{n}$

Example 2

Find the solutions of the quadratic equation $6x^2 + 17x + 7 = 0$ by factorisation.

$6x^2 + 17x + 7 = 0$

$6x^2 + 3x + 14x + 7 = 0$ — Split the x term so that the two coefficients multiply to give ac. $6 \times 7 = 42$ and $3 \times 14 = 42$

$3x(2x + 1) + 7(2x + 1) = 0$ — Factorise the first pair of terms, then the second pair of terms. Take out a factor which is common to both pairs.

$(2x + 1)(3x + 7) = 0$ — Factorise the full expression.

$x = -\frac{7}{3}$ or $x = -\frac{1}{2}$

Sometimes a quadratic will not factorise easily. In these cases you may need to **complete the square**.

Key point

Any quadratic expression can be written in the following way. This is called completing the square.

$$ax^2 + bx + c \equiv a\left(x + \frac{b}{2a}\right)^2 + q$$

You'll need to find the value of q yourself. It will be equal to $c - \frac{b^2}{4a}$

When $a = 1$ and $q = 0$, the expression is known as a **perfect square**. For example, $x^2 + 6x + 9 = (x + 3)^2$

Perfect squares have only one root, so a graph of the quadratic function touches the x-axis only once, at its vertex.

Example 3

By completing the square, find all the solutions of $4 - 3x^2 - 6x = 0$

$3x^2 + 6x - 4 = 0$ — Multiply both sides by -1

$3[x^2 + 2x] - 4 = 0$ — Manipulate the expression to obtain a bracket containing x^2 and the x term.

$3[(x + 1)^2 - 1] - 4 = 0$

$3(x + 1)^2 - 7 = 0$ — Complete the square and expand. Substitute this into the previous equation.

$(x + 1)^2 = \frac{7}{3} \Rightarrow x = -1 + \sqrt{\frac{7}{3}}$ or $-1 - \sqrt{\frac{7}{3}}$

Completing the square is a useful tool for determining the maximum or minimum point of a quadratic function.

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Example 4

Complete the square to determine the minimum point of the graph of $f(x) = 2x^2 + 12x + 16$

$$f(x) = 2x^2 + 12x + 16 = 2(x^2 + 6x) + 16 = 2[(x+3)^2 - 9] + 16$$
$$= 2(x+3)^2 - 2$$

At minimum point $x = -3$

$f(-3) = 2(0)^2 - 2 = -2 \Rightarrow$ minimum point $(-3, -2)$

$(x+3)^2 \geq 0$, so the minimum point is when $(x+3)^2 = 0$

By writing the equation, $ax^2 + bx + c = 0$, $a \neq 0$, in completed square form you can derive the **quadratic formula** for solving equations.

$$ax^2 + bx + c = 0$$
$$a\left[x^2 + \frac{b}{a}x\right] + c = 0$$
$$a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c = 0$$
$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} = \frac{-c}{a}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$
$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Key point

For constants a, b and c, $a > 0$, the solutions to the equation $ax^2 + bx + c = 0$ are

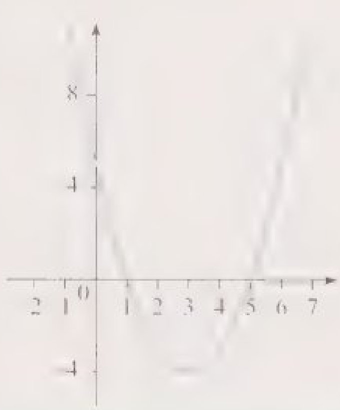
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression inside the square root is called the **discriminant**, Δ

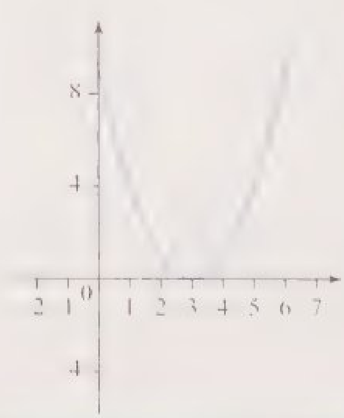
Key point

$\Delta = b^2 - 4ac$

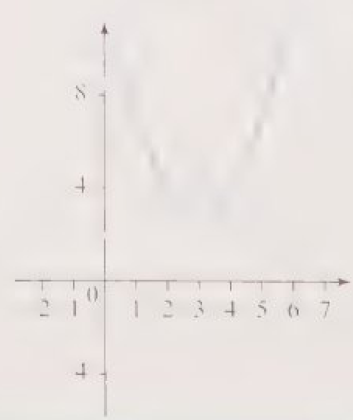
If the discriminant, Δ, is positive, it has two square roots. If Δ is 0, it has one square root. If Δ is negative, it has no real square roots. The value of Δ tells you whether a quadratic equation $ax^2 + bx + c = 0$ has two, one or no real solutions. This result is useful for curve sketching.



If $\Delta > 0$, the quadratic $y = ax^2 + bx + c$ has two distinct roots and the curve crosses the x-axis at two distinct points.



If $\Delta = 0$, the quadratic $y = ax^2 + bx + c$ has one repeated root and the x-axis is a tangent to the curve at this point.



If $\Delta < 0$, the quadratic $y = ax^2 + bx + c$ has no (real) roots and the curve does not cross the x-axis at any point.

Example 5

Use the discriminant $\Delta = b^2 - 4ac$ to determine how many roots each of these quadratic equations have.

a $x^2 + 2x + 1 = 0$ **b** $x^2 + 2x - 8 = 0$ **c** $x^2 + 6x + 10 = 0$

a $\Delta = b^2 - 4ac = 2^2 - 4 \times 1 \times 1 = 0$

So $x^2 + 2x + 1 = 0$ has one repeated root.

b $\Delta = b^2 - 4ac = 2^2 - 4 \times 1 \times -8 = 36 > 0$

So $x^2 + 2x - 8 = 0$ has two distinct roots.

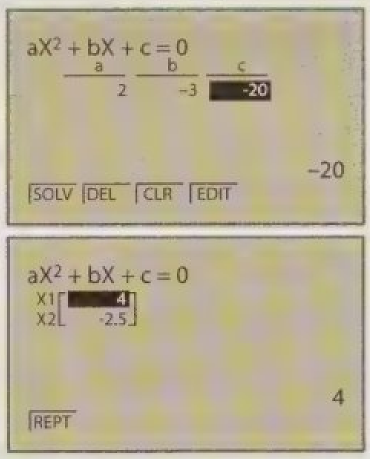
c $\Delta = b^2 - 4ac = 6^2 - 4 \times 1 \times 10 = -4 < 0$

So $x^2 + 6x + 10 = 0$ has no real roots.

Calculator

Try it on your calculator

You can solve a quadratic equation on a calculator.



Activity

Find out how to solve $2x^2 - 3x - 20 = 0$ on *your* calculator.

Exercise 1.4A Fluency and skills

1 Solve these quadratic equations by factorisation.

a $x^2 - 18 = 0$ **b** $2x^2 - 6 = 0$

c $4x^2 + 5x = 0$ **d** $x^2 + 2\sqrt{3}x + 3 = 0$

e $2x^2 + 5x - 3 = 0$ **f** $3x^2 - 23x + 14 = 0$

g $16x^2 - 24x + 9 = 0$ **h** $18 + x - 4x^2 = 0$

2 For each quadratic function

i Factorise the equation,

ii Use your answer to part **i** to sketch a graph of the function.

a $f(x) = x^2 + 3x + 2$ **b** $f(x) = x^2 + 6x - 7$

c $f(x) = -x^2 - x + 2$ **d** $f(x) = -x^2 - 7x - 12$

e $f(x) = 2x^2 - x - 1$ **f** $f(x) = -3x^2 + 11x + 20$

3 Solve these quadratic equations using the formula. Write your answers both exactly (in surd form) and also, where appropriate, correct to 2 decimal places.

a $3x^2 + 9x + 5 = 0$ **b** $4x^2 + 5x - 1 = 0$

c $x^2 + 12x + 5 = 0$ **d** $28 - 2x - x^2 = 0$

e $x^2 + 15x - 35 = 0$ **f** $34 + 3x - x^2 = 0$

g $4x^2 - 36x + 81 = 0$ **h** $3x^2 - 23x + 21 = 0$

i $5x^2 + 16x + 9 = 0$ **j** $10x^2 - x - 1 = 0$

4 For each quadratic function, complete the square and thus determine the coordinates of the minimum or maximum point of the curve.

a $f(x) = x^2 - 14x + 49$ **b** $f(x) = x^2 + 2x - 5$

c $f(x) = -x^2 - 6x - 5$ **d** $f(x) = -x^2 + 4x + 3$

e $f(x) = 9x^2 - 6x - 5$ **f** $f(x) = -2x^2 - 28x - 35$

5 Complete the square to work out the exact solutions to these quadratic equations.

a $x^2-2x=0$ b $3-4x-x^2=0$
c $x^2-14x+33=0$ d $x^2+8x+10=0$
e $x^2-6x+9=0$ f $x^2+10x+24=0$
g $x^2+22x+118=0$ h $x^2-16x+54=0$
i $4x^2-12x+2=0$ j $9x^2+12x-2=0$
k $x^2+11x+3=0$ l $9x^2-30x-32=0$

6 Solve these quadratic equations using your calculator.

a $2x^2-6x=0$ b $x^2+2x-15=0$
c $x^2-5x-6=0$ d $8+2x-x^2=0$
e $2x^2-x-15=0$ f $6+8x-8x^2=0$

7 By evaluating the discriminant, identify the number of real roots of these equations.

a $x^2+2x-5=0$ b $13+3x-x^2=0$
c $x^2+5x+5=0$ d $-3+2x-x^2=0$
e $4x^2+12x+9=0$ f $-35+2x-x^2=0$
g $9x^2-66x+121=0$
h $-100-100x-100x^2=0$

Reasoning and problem-solving

Strategy

To solve a problem involving a quadratic curve

1. Factorise the equation or complete the square and solve as necessary.
2. Sketch the curve using appropriate axes and scale.
3. Mark any relevant points in the context of the question.

Example 6

The motion of a body, which has an initial velocity u and acceleration a, is given by the formula $s=ut+\frac{1}{2}at^2$, where s is the displacement after a time t

a By completing the square and showing all intermediate steps, sketch the graph of s against t when $u=8$ and $a=-4$

b What is happening to the body at the turning point of the graph?

a $s=8t-2t^2$
$=-2(t^2-4t)$
$=-2[(t-2)^2-4]=-2(t-2)^2+8$

So there is a turning point at (2, 8)

Where the curve crosses the x-axis,

$2t(4-t)=0$
$t=0$ or 4

Therefore the curve cuts the x-axis at (0, 0) and (4, 0)

b The body is reversing direction. At this point it has zero velocity.

1 Factorise and complete the square.

2 3 Sketch the curve and mark on it any relevant points.

Exercise 1.4B Reasoning and problem-solving

1 A designer is lining the base and sides of a rectangular drawer, dimensions $2x$ cm by $3x$ cm by 5 cm, with paper.

The total area of paper is 4070 cm^2.

a Write and solve an equation to find x

b Hence work out the volume of the drawer in litres.

2 Sam and his mother Jane were both born on January 1st.

In 2002, Sam was x years old and Jane was $2x^2+11x$ years old. In 2007, Jane was five times as old as Sam.

a Form and solve a quadratic equation in x

b Hence work out Jane's age when Sam was born.

3 A photo is to be pasted onto a square of white card with side length x cm. The photo is $\frac{3}{4}x$ cm long and its width is 20 cm less than the width of the card. The area of the remaining card surrounding the photo is 990 cm^2. Work out the dimensions of the card and photo.

4 A piece of wire is bent into a rectangular shape with area 85 in^2.

The total perimeter is 60 inches and the rectangle is x^2 in long.

a Form an expression for the area of the rectangle.

b By substituting z for x^2, form a quadratic equation in z

c Hence work out all possible values of x

5 A man stands on the edge of a cliff and throws a stone out over the sea. The height, h m, above the sea that the stone reaches after t seconds is given by the formula $h=50+25t-5t^2$

a Complete the square.

b Sketch the graph of $h=50+25t-5t^2$

c Use your graph to estimate

i The maximum height of the stone above the sea and the time at which it reaches this height,

ii The time when the stone passes the top of the cliff on the way down,

iii The time when the stone hits the sea.

6 A firm making glasses makes a profit of y thousand pounds from x thousand glasses according to the equation $y=-x^2+5x-2$

a Sketch the curve.

b Use your graph to estimate

i The value of x to give maximum profit,

ii The value of x not to make a loss,

iii The range of values of x which gives a profit of more than £3250

7 The mean braking distance, d yards, for a car is given by the formula $d=\frac{v^2}{50}+\frac{v}{3}$, where v is the speed of the car in miles per hour.

a Sketch this graph for $0\le v\le 80$

b Use your sketch to estimate the safe braking distance for a car driving at

i 15 mph ii 45 mph iii 75 mph

c A driver just stops in time in a distance of 50 yards.
How fast was the car travelling when the brakes were applied?

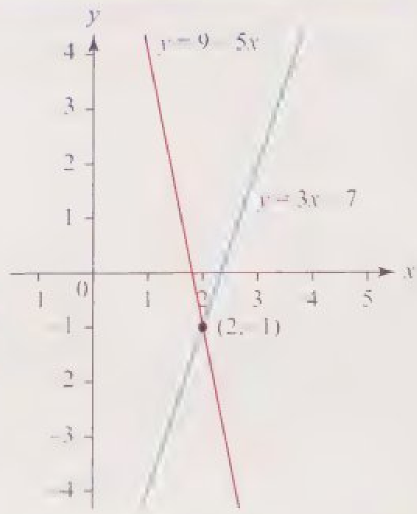
Challenge

8 Use a suitable substitution to solve $2(k^6-11k^3)=160$ for k. Give your answers in exact form.

1.5 Simultaneous equations

Fluency and skills

The graphs of $3x - y = 7$ and $5x + y = 9$ are shown.



Only one pair of values for (x, y) satisfies both equations. This corresponds to the point of intersection of the two graphs. In this example it is $x = 2$ and $y = -1$

When you solve two equations together like this, they are called **simultaneous equations.**

You can solve a pair of linear simultaneous equations **Key point**

1. Graphically.
2. By eliminating one of the **variables.**
3. By substituting an expression for one of the variables from one equation into the other.

Example 1

Solve the simultaneous equations $2a - 5b = -34$ and $3a + 4b = -5$ by elimination.

$2a - 5b = -34$ (1) and $3a + 4b = -5$ (2) — Label the equations 1 and 2

$8a - 20b = -136$ and $15a + 20b = -25$ — Multiply equation (1) by four and equation (2) by five.

$23a = -161$ — Add equations to eliminate the b terms

$a = -7$

$-14 - 5b = -34$ — To find b, substitute $a = -7$ into equation (1)

$-14 + 34 = 5b \Rightarrow b = 4$

$a = -7$ and $b = 4$ — You can check your answers by solving the simultaneous equations on your calculator

Example 2

Solve the simultaneous equations $2c - 3d = 8$ and $4c + 5d = 5$ by substitution.

$2c - 3d = 8$ (1)

$4c + 5d = 5$ (2) — Label the equations 1 and 2

$c = \frac{8 + 3d}{2}$ — Rearrange equation (1) to make c the subject

so $4\left(\frac{8 + 3d}{2}\right) + 5d = 5$ — Substitute for c in equation (2) and solve to find d

$2(8 + 3d) + 5d = 5 \Rightarrow d = -1$

$2c - 3(-1) = 8 \Rightarrow c = \frac{5}{2}$ — To find c, substitute $d = -1$ into equation (1)

So the solution is $c = \frac{5}{2}$ and $d = -1$

A straight line can intersect a quadratic curve at either two, one or zero points.

You should equate expressions for the curve and the line to find the points of intersection. You will then obtain a quadratic equation in the form $ax^2 + bx + c = 0$

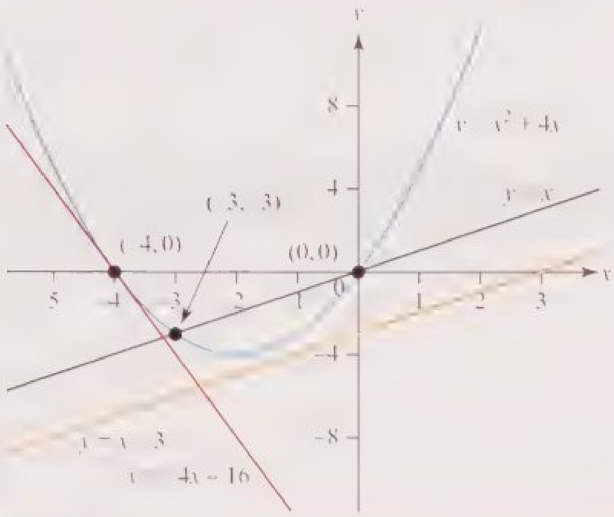
You can use the discriminant, $\Delta = b^2 - 4ac$, to show if this equation has two, one or no solutions.

The diagram shows the graphs of $y = x^2 + 4x$, $y = -4x - 16$, $y = x$ and $y = x - 3$

$y = -4x - 16$ touches $y = x^2 + 4x$ at the point $(-4, 0)$

$y = x$ intersects $y = x^2 + 4x$ at the points $(0, 0)$ and $(-3, -3)$

$y = x - 3$ does not intersect $y = x^2 + 4x$ at any point.



ICT Resource online

To investigate simultaneous equations, click this link in the digital book.

Example 3

Solve the simultaneous equations $y = x^2 + 4x$ and $y + 4x + 16 = 0$. Interpret your answers graphically.

$y = -4x - 16$ — Make y the subject.

$x^2 + 4x = -4x - 16$ — Substitute $y = -4x - 16$ into the quadratic.

$x^2 + 8x + 16 = 0 \Rightarrow (x+4)(x+4) = 0 \Rightarrow$ so $x = -4$ — Solve the resulting quadratic equation.

When $x = -4$, $y = -4 \times -4 - 16 = 0$

The solution is $x = -4$, $y = 0$

$\therefore$ the straight line $y + 4x + 16 = 0$ touches the curve $y = x^2 + 4x$ at a single point $(-4, 0)$, and is therefore a tangent to the curve. — Derive the nature of the roots of the quadratic equation.

Exercise 1.5A Fluency and skills

Solve the simultaneous equations from 1 to 19. You must show your working.

1 $x + y = 7$, $2x + y = 11$

2 $a + b = 7$, $2a - b = 11$

3 $2a + 3b = 8$, $a + 2b = 5$

4 $4x + 2y = 2$, $5x - 2y = 7$

5 $4c + 2d = 2$, $5c + 2d = 7$

6 $2e - f = 13$, $e - f = 5$

7 $2e - f = 13$, $e + f = 5$

8 $7g + 4h = 12$, $-5g + 4h = 12$

9 $7x + 4y = 12$, $-5x - 4y = 12$

10 $3m - 4n = -15$, $-3m - n = 0$

11 $-3m - 4n = -15$, $-3m - n = 0$

12 $4a - 21 = b$, $2b = 13 - 3a$

13 $5c + 2d = 9$
$3c = d - 10$

14 $3c - 4d = 29$
$4c = 13 + 3d$

15 $2e = 13 - 3f$
$6f = e - 4$

16 $x + y = 3$
$x^2 + y = 3$

17 $g + h = 1$
$g^2 - h = 5$

18 $3g + 2h = 13$
$h + 2g^2 = 20$

19 $10m = 7n + 17$
$m = n^2$

20 Find the point(s) of intersection of the graphs $y^2 = 5x$ and $y = x$. Show your working.

21 Find the point(s) of intersection of the graphs $y^2 = 6x + 7$ and $y = x + 2$. Show your working.

22 Solve the simultaneous equations $x + y^2 = 2$ and $2 = 3x + y$, showing your working. Find the points of intersection.

23 Solve the simultaneous equations $y^2 = -1 - 5x$, $y = 2x + 1$. Find the points of intersection, showing your working.

24 A curve has equation $xy = 20$

A straight line has equation $y = 8 + x$

Solve the two equations simultaneously and show that the points of intersection are $(2, 10)$ and $(-10, -2)$

Reasoning and problem-solving

Strategy

To solve a simultaneous equations problem

1. Use the information in the question to create the equations.
2. Use either elimination or substitution to solve your equations.
3. Check your solution and interpret it in the context of the question.

Example 4

A rectangle has sides of length $(x + y)$ m and $2y$ m. The rectangle has a perimeter of 64 m and an area of 240 m². Calculate the possible values of x and y. Show your working.

The perimeter is $2[x + y + 2y] = 2x + 6y$
and the area is $2y(x + y) = 2xy + 2y^2$

$\therefore 2x + 6y = 64$ and $2xy + 2y^2 = 240$ — (1) Create the equations.

$x + 3y = 32$ or $x = 32 - 3y$ (1)

$2xy + 2y^2 = 240$ or $xy + y^2 = 120$ (2)

$(32 - 3y)y + y^2 = 120$ — Substitute $x = 32 - 3y$ from equation (1) into equation (2)

$32y - 2y^2 = 120$

$2y^2 - 32y + 120 = 0$

$y^2 - 16y + 60 = 0$

$(y - 6)(y - 10) = 0$

$y = 6$ or 10

$x = 32 - 3 \times 6 = 14$ or $32 - 3 \times 10 = 2$ — (2) Substitute the values for y into equation (1) to obtain values for x

Checking $14 \times 6 + 6^2 = 84 + 36 = 120$ ✓
or $2 \times 10 + 10^2 = 20 + 100 = 120$ ✓

$\therefore$ the two possible values of x and y are
$x = 14, y = 6$ or $x = 2, y = 10$ — (3) Check your solution in equation (2) and interpret it in context.

Exercise 1.5B Reasoning and problem-solving

1 In a recent local election, the winning candidate had an overall majority of 257 votes over her only opponent. There were 1619 votes cast altogether.

Form a pair of simultaneous linear equations.

How many votes did each candidate poll?

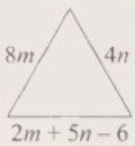
2 A fisherman is buying bait. He can either buy 6 maggots and 4 worms for £1.14 or 4 maggots and 7 worms for £1.28

How much do maggots and worms cost individually? Show your working.

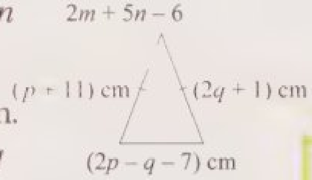
3 The straight line $y = mx + c$ passes through the points $(3, -10)$ and $(-2, 5)$

Find the values of m and c

4 This triangle is equilateral.
Find the values of m and n



5 This triangle is isosceles.
It has a perimeter of 150 cm.
Find the values of p and q



6 The ages of Florence and Zebedee are in the ratio 2 : 3
In 4 years' time, their ages will be in the ratio 3 : 4
Use simultaneous equations to calculate how old they are now. Show your working.

7 **a** Try to solve the simultaneous linear equations $y - 2x = 3$ and $4x = 2y - 6$

How many solutions are there? Explain your answer.

b Try to solve the simultaneous linear equations $y - 2x = 3$ and $4x = 2y - 8$

How many solutions are there? Explain your answer.

8 The equations of three straight lines and a parabola are $y + 2x + 4 = 0$, $y + 11x - 27 = 0$, $x - y + 3 = 0$ and $y = 2x^2 - 19x + 35$. One of the lines intersects the curve at two points, one 'misses' the curve and one is a tangent to the curve. Investigate the nature of the relationship between each of these lines and the curve, and calculate any real points of intersection.

9 Prove that the line $y = 2x - 9$ does not intersect the parabola $y = x^2 - x - 6$

10 The sums of the first n terms of two sequences of numbers are given by formulae $S_1 = 2n + 14$ and $2S_2 = n(n + 1)$. For which values of n does $S_1 = S_2$? Explain your results carefully.

11 A farmer has 600 m of fencing. He wants to use it to make a rectangular pen of area 16 875 m^2

Calculate the possible dimensions of this pen.

12 The equation $x^2 + y^2 = 25$ represents a circle of radius 5 units. Prove that the line $3x + 4y = 25$ is a tangent to this circle and find the coordinates of the point where the tangent touches the circle.

13 An ellipse has the equation $4y^2 + 9x^2 = 36$

Show that the line $y = 2x + 1$ intersects this ellipse at the points $\left(\frac{-8 \pm 12\sqrt{6}}{25}, \frac{9 \pm 24\sqrt{6}}{25}\right)$

Challenge

14 Two particles, A and B, move along a straight line. At a time, t, the position of A from a fixed point, O, on the line is given by the formula $x = 2 + 8t - t^2$ and that of B by $x = 65 - 8t$

a How far from O is each particle initially?

b Explain how you know that B is initially moving towards O

c Explain how you know A moves away from O and then moves back towards O

d What is the maximum distance of A from O?

e Calculate the first time when both particles are at the same distance from O

f In which directions are A and B moving at the time you calculated in part **e**?

1.6 Lines and circles

Fluency and skills

The equation of a straight line can be written in the form $y = mx + c$

where m is the gradient and c is the y-intercept.

Key point

A straight line can also be written in the form
$y - y_1 = m(x - x_1)$
where (x_1, y_1) is a point on the line and m is the gradient.

You can rearrange the general equation of a straight line to get a formula for the gradient.

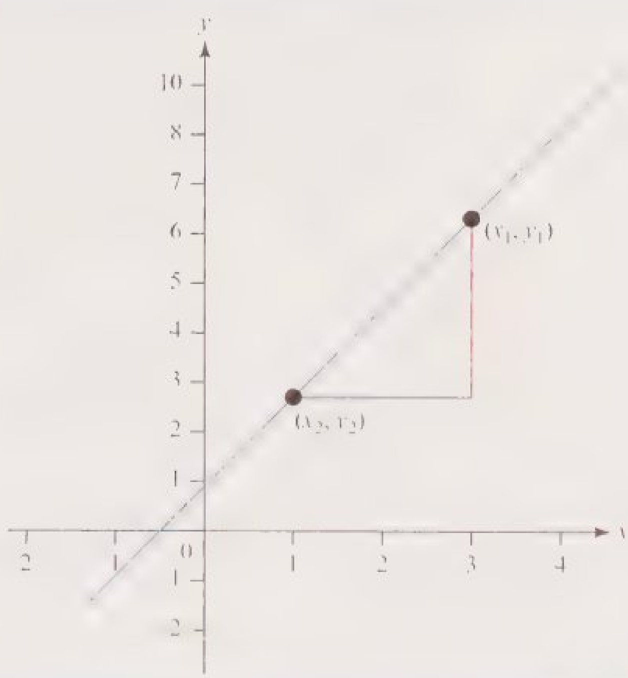
Key point

The gradient of a straight line through two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

You can use Pythagoras' theorem to find the distance between two points.

Key point

The distance between two points (x_1, y_1) and (x_2, y_2) is given by the formula $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$



Key point

The coordinates of the midpoint of the line joining (x_1, y_1) and (x_2, y_2) are given by the formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

PURE

You can use the gradients of two lines to decide if they are **parallel** or **perpendicular**.

Key point

Two lines are described by the equations
$y_1 = m_1x + c_1$ and $y_2 = m_2x + c_2$

If $m_1 = m_2$, the two lines are parallel.

If $m_1 \times m_2 = -1$, the two lines are perpendicular.

Example 1

A straight line segment joins the points $(-2, -3)$ and $(4, 9)$

a Work out the midpoint of the line segment.

b Work out the equation of the perpendicular bisector of the line segment.
Give your answer in the form $ay + bx + c = 0$ where a, b and c are integers.

a $\left(\frac{-2+4}{2}, \frac{-3+9}{2}\right) = (1, 3)$

b Gradient of line segment $= \frac{9--3}{4--2} = 2$ — Use gradient $= \frac{y_2 - y_1}{x_2 - x_1}$

Gradient of perpendicular bisector $= -\frac{1}{2}$ — Use $m_1 \times m_2 = -1$

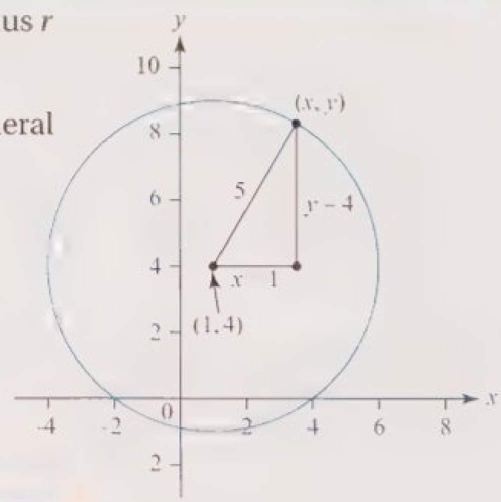
$y - 3 = -\frac{1}{2}(x - 1)$ — Use $y - y_1 = m(x - x_1)$

$2y - 6 = -x + 1$

$2y + x - 7 = 0$ — Multiply through by 2 and rearrange to the required form.

On a graph, the equation of any circle with centre (a, b) and radius r has the same general form.

The diagram shows a circle, centre $(1, 4)$ and radius 5, with a general point (x, y) shown on the circumference.



The vertical distance of the point (x, y) from the centre is $y - 4$ and the horizontal distance of the point (x, y) from the centre is $x - 1$.

Using Pythagoras' theorem for the right-angled triangle shown, you get $(x - 1)^2 + (y - 4)^2 = 5^2$

Notice that this equation is in the form $(x - a)^2 + (y - b)^2 = r^2$
This is the equation for any circle.

Key point

The equation of a circle, centre (a, b) and radius r, is
$$(x - a)^2 + (y - b)^2 = r^2$$

Key point

For a circle centred at the origin, $a = 0$ and $b = 0$, so the equation of the circle is simply
$$x^2 + y^2 = r^2$$

Example 2

Work out the equation of the circle with centre $(-4, 9)$, radius $\sqrt{8}$

Write your answer without brackets.

$(x+4)^2+(y-9)^2=8$ — Use $(x-a)^2+(y-b)^2=r^2$

$x^2+8x+16+y^2-18y+81-8=0$

$x^2+y^2+8x-18y+89=0$

Example 3

Work out the centre and radius of the circle $4x^2-4x+4y^2+3y-6=0$

$4x^2-4x+4y^2+3y-6=0$

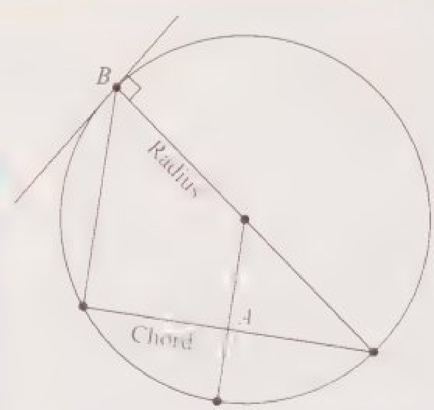
$x^2-x+y^2+\frac{3}{4}y-\frac{3}{2}=0$ — Divide by 4

$\left(x-\frac{1}{2}\right)^2-\frac{1}{4}+\left(y+\frac{3}{8}\right)^2-\frac{9}{64}-\frac{3}{2}=0$ — Complete the square twice using both x and y terms.

$\left(x-\frac{1}{2}\right)^2+\left(y+\frac{3}{8}\right)^2=\frac{121}{64}$ — Use $(x-a)^2+(y-b)^2=r^2$

Hence the circle has centre $\left(\frac{1}{2},-\frac{3}{8}\right)$ and radius $\frac{11}{8}$ — You can check by drawing the circle on a graphics calculator.

When you're working with equations of circles, it's useful to remember some facts about the lines and angles in a circle. You should have come across these before in your studies.



Key point

- If a triangle passes through the centre of the circle, and all three corners touch the circumference of the circle, then the triangle is right-angled.
- The perpendicular line from the centre of the circle to a chord bisects the chord (Point A in the diagram).
- Any tangent to a circle is perpendicular to the radius at the point of contact (B).

Exercise 1.6A Fluency and skills

1 a Write down the equation of the straight line with gradient $-\frac{2}{3}$ that passes through the point $(-4, 7)$. Give your answer in the form $ax+by+c=0$, where a, b and c are integers.

b Does the point $(13, 3)$ lie on the line described in part **a**?

2 Find the gradient and y-intercept of the line $4x-3y=8$

3 a Show that the lines $2x-3y=4$ and $6x+4y=7$ are perpendicular.

b Show that the lines $2x-3y=4$ and $8x-12y=7$ are parallel.

4 Write down the gradient and y-intercept of the line $\frac{2}{3}x+\frac{3}{4}y+\frac{7}{8}=0$

5 Calculate the gradient of the straight-line segment joining the points $(-5,-6)$ and $(4,-1)$

Hence write down the equation of the line.

6 Write, in both the form $y=mx+c$ and the form $ax+by+c=0$, the equation of the line with gradient -3 passing through $(-8,-1)$

7 Work out the midpoint and length of the line segment joining each of these pairs of points.

a $(2,2)$ and $(6,10)$

b $(-3,-4)$ and $(2,-3)$

c $(0,0)$ and $(\sqrt{5},2\sqrt{3})$

8 Which of these lines are parallel or perpendicular to each other?

$2x+3y=4$ $\quad 4x-5y=6$ $\quad y=4x+8$
$10x-8y=5$ $\quad 10x+8y=5$ $\quad 3y-12x=7$
$6x+9y=12$

9 a Write down the equation of the straight line through the point $(5,-4)$ which is parallel to the line $2x+3y-6=0$

b Write down the equation of the straight line through the point $(-2,-3)$ which is perpendicular to the line $3x+6y+5=0$

10 Write down the equations of each of these circles.

Expand your answers into the form $ax^2+bx+cy^2+dy+e=0$

a Centre $(1,8)$; radius 5

b Centre $(6,-7)$; radius 3

c Centre $(\sqrt{5},\sqrt{2})$; radius $\sqrt{11}$

11 Work out the centre and radius of each of these circles.

a $x^2+18x+y^2-14y+30=0$

b $x^2+12x+y^2+10y-25=0$

c $x^2-2\sqrt{3}x+y^2+2\sqrt{7}y-1=0$

12 Prove that the points $A(-10,-12)$, $B(6,18)$ and $C(-2,-14)$ lie on a semicircle.

13 Write the equation of the tangent to the circle with centre $(4,-3)$ at the point $P(-2,-1)$

14 $(-3,9)$ is the midpoint of a chord within a circle with centre $(7,-1)$ and radius 18

a Calculate the equation of the circle.

b Calculate the length of the chord.

c Complete the square to find the exact coordinates of the ends of the chord.

15 Write down the equations of each of the circles with diameters from

a $(0,0)$ to $(0,20)$

b $(2,6)$ to $(6,2)$

c $(4,-2)$ to $(-3,16)$

d $(-4,-5)$ to $(-\sqrt{2},\sqrt{5})$

16 The circle with equation $x^2+y^2=25$ crosses the line $y=7-x$ at two points. Solve these simultaneous equations and find the points of intersection. Show your working.

17 a Write down the equation of the straight line with gradient $-\frac{1}{2}$ that passes through the point $(1,1)$

b Write down the equation of the circle with radius 3 and centre $(2,2)$

c The line in part **a** crosses the circle in part **b** at two points. Solve these simultaneous equations and find the coordinates of these two points. Show your working.

18 a Write down the equation of the straight line that passes through the points $(3,5)$ and $(-1,-3)$

b Write down the equation of the circle with centre $(1,0)$ and radius $\frac{17}{2}$

c The line in part **a** crosses the circle in part **b** at two points. Solve these simultaneous equations and find the coordinates of these two points. Show your working.

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Reasoning and problem-solving

Strategy

To solve a problem involving a straight line or a circle

1. Choose the appropriate formulae.
2. Apply any relevant rules and theorems. Draw a sketch if it helps.
3. Show your working and give your answer in the correct form.

Example 4

a A diagonal of a rhombus has equation $2x - 3y + 8 = 0$ and midpoint $(-3, 7)$

Work out the equation of the other diagonal.

b One vertex of the rhombus on the original diagonal is $(14, 12)$

Work out the coordinates of the opposite vertex.

a $y = \frac{2}{3}x + \frac{8}{3}$ — (1) Write the equation of the diagonal in the form $y = mx + c$

Gradient of this diagonal is $\frac{2}{3}$

$\therefore$ gradient of the other diagonal is $-\frac{3}{2}$ — (2) Diagonals of a rhombus are perpendicular so $m_1 \times m_2 = -1$

Hence the equation of other diagonal is

$y - 7 = -\frac{3}{2}(x + 3)$, so — (1) Use $y - y_1 = m(x - x_1)$

$2y + 3x - 5 = 0$

b The vertex of the diagonal is $(14, 12)$ and the midpoint is $(-3, 7)$

so $(-3, 7) = \left(\frac{x_1 + 14}{2}, \frac{y_1 + 12}{2}\right)$ — (1) Use midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

so $x_1 = -20$ and $y_1 = 2$

$\therefore$ the other vertex is $(-20, 2)$

Example 5

$A(-7, 1)$, $B(11, 13)$ and $C(19, 1)$ are three points on a circle. Prove that AC is a diameter.

Gradient of $AB = \left(\frac{13 - 1}{11 + 7}\right) = \frac{2}{3}$

Gradient of $BC = \left(\frac{1 - 13}{19 - 11}\right) = -\frac{3}{2}$ — (1) Use gradient $= \frac{y_2 - y_1}{x_2 - x_1}$

$\frac{2}{3} \times -\frac{3}{2} = -1$ so AB is perpendicular to BC — (2) Use $m_1 \times m_2 = -1$ to prove that the lines are perpendicular.

Therefore ABC is 90° and, since the angle in a semicircle is a right angle, ABC is a semicircle and thus AC is a diameter. — (2) (3) Use angle in a semicircle theorem and answer the question.

Exercise 1.6B Reasoning and problem-solving

1 A quadrilateral has vertices $P(-15,-1)$, $Q(-3,4)$, $R(12,12)$ and $S(0,7)$. Write the equation of each side and identify the nature of the quadrilateral.

2 A giant kite is constructed using bamboo for the edges and diagonals and card for the sail. When mapped on a diagram, the ends of the long diagonal are at $P(2,-2)$ and $R(-14,-14)$ and the diagonals intersect at M. The short diagonal QS divides RP in the ratio 3 : 1 and $MP = MQ = MS$

Calculate

a The coordinates of M,

b The coordinates of Q and S,

c The equations of the diagonals,

d The equations of the sides of the kite,

e The area of card needed to make the kite,

f The total length of bamboo required for the structure.

3 On a map, three villages are situated at points $A(2,-5)$, $B(10,1)$ and $C(9,-6)$, and all lie on the circumference of a circle.

a Find the equations of the perpendicular bisectors of AB and AC

b Hence work out the centre and equation of the circle and show that the triangle formed by the villages is right-angled.

4 The equation of a circle, centre C, is $x^2 + y^2 - 4x - 12y + 15 = 0$

a Prove the circle does not intersect the x-axis.

b P is the point $(8, 1)$. Find the length CP and determine whether P lies inside or outside the circle.

c Write the set of values of k for which $3y - 4x = k$ is a tangent to the circle.

5 A circle, centre C, has equation $x^2 + y^2 - 20x + 10y + 25 = 0$ and meets the y-axis at Q. The tangent at $P(16, 3)$ meets the y-axis at R. Work out the area of the triangle PQR

6 A park contains a circular lawn with a radius of 50 m. If the park is mapped on a set of axes, with the y-axis due north, this lawn is centred on the point $(-40,-80)$

A straight, underground water pipe runs through the park and its position is represented by the line $y + 2x = -60$

The town council wants to install a drinking fountain in the park. The fountain must be directly above the underground pipe, it must lie on the outer edge of the lawn, and it must be as close to the east side of the park as possible.

Determine the coordinates of the only possible location for the new drinking fountain.

Challenge

7 One diagonal of a rhombus has equation $2y - x = 20$. The two corners that form the other diagonal in the rhombus touch the edges of a circle with equation $x^2 + y^2 - 8x - 24y + 144 = 0$

a Find the radius of the circle.

b Find the equation of the other diagonal of the rhombus.

8 a Show that if (a, c) and (b, d) are the ends of a diameter of a circle, the equation of the circle is

$$(x-a)(x-b) + (y-c)(y-d) = 0$$

b The line segment with endpoints $(-3, 12)$ and $(13, 0)$ is the diameter of a circle. Work out the equation of the circle. Give your answer without brackets.

1.7 Inequalities

Fluency and skills

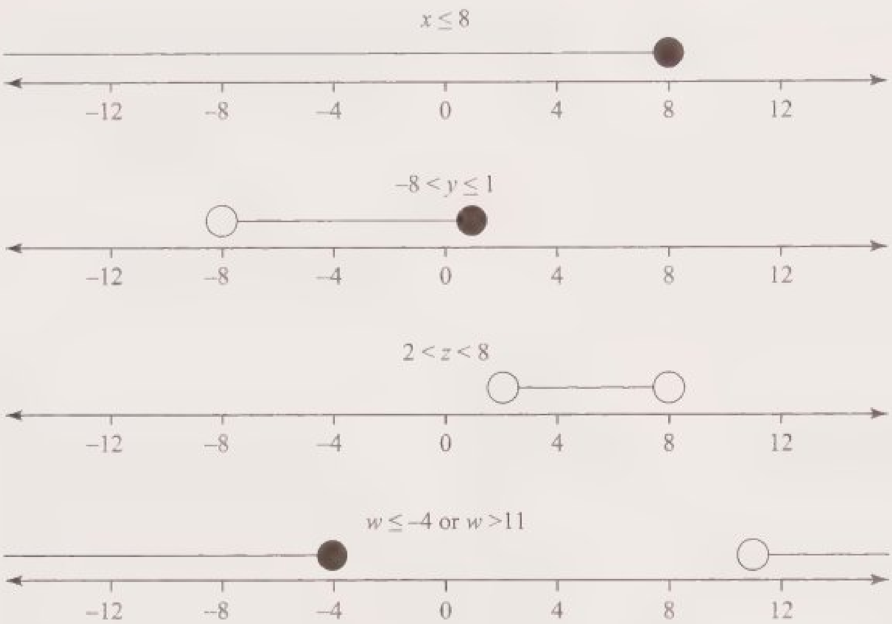
See p.300 For a list of mathematical notation

You can express **inequalities** using the symbols < (less than), > (greater than), ≤ (less than or equal to) and ≥ (greater than or equal to).

Key point

You can represent inequalities on a number line.

For example



On a number line, you use a dot, ●, when representing ≤ or ≥, and you use an empty circle, ○, when representing < or >

You can also use set notation to represent inequalities.

For example, the last inequality could be represented in any of the following ways.

- $w \in \{w: w \leq -4 \text{ or } w > 11\}$
  w is an element of the set of values that are less than or equal to −4 or greater than 11
- $w \in \{w: w \leq -4\} \cup \{w: w > 11\}$
  w is an element of the union of two sets. This means w is in one set or the other.
- $w \in (-\infty, -4] \cup (11, \infty)$
  w is in the union of two intervals. Square brackets indicate the end value is included in the interval, round brackets indicate that the end value is not included in the interval.

To solve **linear inequalities** you follow the same rules for solving linear equations, but with one exception.

Key point

When you multiply or divide an inequality by a negative number, you reverse the inequality sign.

Example 1

Solve the inequality $4(3z+12) \le 5(4z-8)$

$4(3z+12) \le 5(4z-8)$

$12z+48 \le 20z-40$ — Expand the brackets.

$12z-20z \le -40-48$ — Subtract $20z$ and 48 from each side.

$-8z \le -88$

$z \ge 11$ — Divide by −8. Remember to reverse the inequality sign.

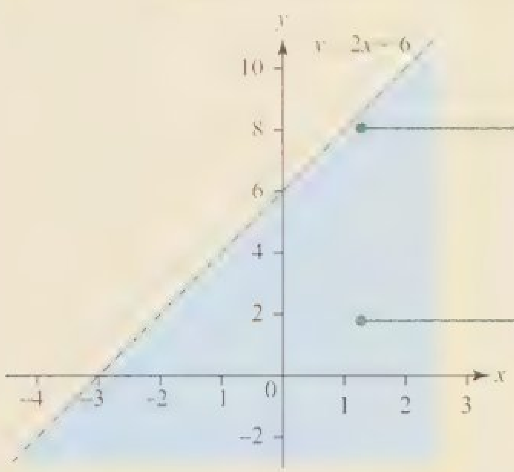
Example 2

Shade each of these regions on a graph.

a $y-2x<6$

b $y+3x \le 8$; $y-2x<4$; $y>0$

a $y-2x<6$ — Sketch the line $y-2x=6$



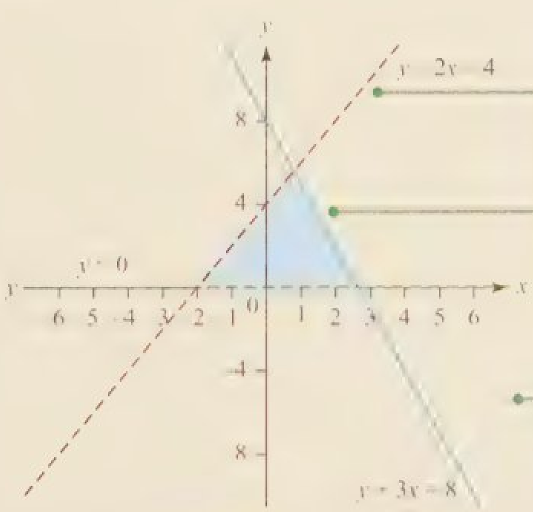
Use a dashed line to represent < or >

Test a point on one side of the line $y-2x=6$ and shade the region that is needed.

b $y+3x \le 8$

$y-2x<4$

$y>0$



Test points and shade the correct region.

Use a solid line to represent ≤ or ≥

You can check your sketches using a graphics calculator. Use the graphing function and select the appropriate inequality symbol.

A **quadratic inequality** looks similar to a quadratic equation except it has an inequality sign instead of the '=' sign.

You can solve quadratic inequalities by starting the same way you would to solve quadratic equations. The answer, however, will be a range of values rather than up to two specific values.

Example 3

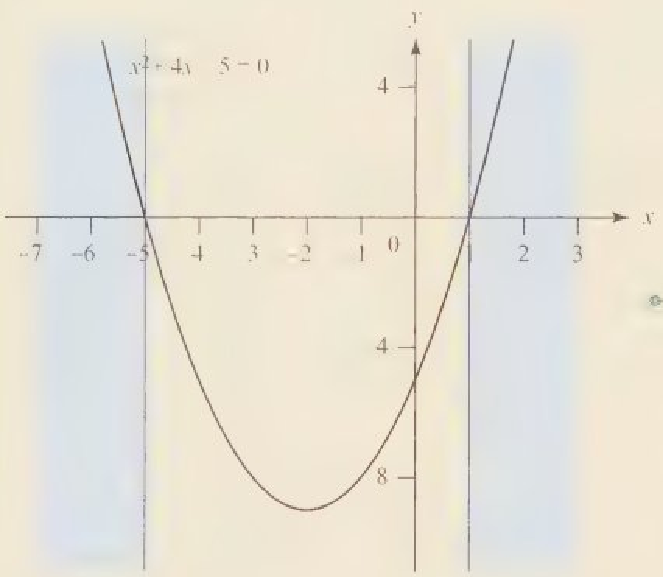
Solve the equation $x^2 + 4x - 5 = 0$ and sketch the graph $y = x^2 + 4x - 5$

Use your sketch to solve the inequality $x^2 + 4x - 5 \geq 0$

$x^2 + 4x - 5 = 0$

$(x - 1)(x + 5) = 0$ — Factorise

so $x = 1$ or -5



Look at the range of values for x for which $(x - 1)(x + 5) \geq 0$

These are the values for which the curve $y = (x - 1)(x + 5)$ is on or above the x-axis.

The shaded regions on the graph show the solution to the inequality.

The solution is $x \geq 1$ or $x \leq -5$

In this case, x could lie in the first region *or* the second. It cannot lie in both so the answer must use the word 'or'.

You could also have solved the question in Example 3 using the factorised form, by considering signs.

The product of the two brackets is positive if they are both positive or both negative.

$x - 1 \geq 0$ *and* $x + 5 \geq 0$ only if $x \geq 1$

$x - 1 \leq 0$ *and* $x + 5 \leq 0$ only if $x \leq -5$

The solution is $x \geq 1$ or $x \leq -5$

Exercise 1.7A Fluency and skills

1 Show the following inequalities on a number line.

a $s \geq 14$

b $3 < u \leq 9$

c $v < 5$ and $v > 14$

d $r > 7$ and $r \leq 12$

2 Draw graphs to show these inequalities. You can check your sketches using a graphics calculator.

a $x > -4$

b $y \geq 5$

c $y + x < 6$

d $2y - 3x < 5$

e $3y + 4x \leq 8$

f $2y > 10 - 4x$

g $y < x + 4; y + x + 1 > 0; x \leq 5$

h $y \geq 2;\; x + y < 7;\; y - 2x - 4 \leq 0$

3 Find the values of x for which

a $2x - 9 > -6$

b $15 - 2x \geq 8x + 34$

c $2(4x - 1) + 6 < 15 - 3x$

d $3(x - 3) + 6(5 - 4x) \leq 54$

e $4(2x + 1) - 7(3x + 2) > 5(4 - 2x) - 6(3 - x)$

f $4\left(3x - \frac{1}{2}\right) + 2(8 - 3x) < 6\left(x + \frac{3}{2}\right) - 2\left(x - \frac{5}{2}\right)$

4 For each part **a** to **h**, sketch a suitable quadratic graph and use your sketch to solve the given inequality.

a $x^2 + x - 6 > 0$

b $x^2 + 11x + 28 < 0$

c $x^2 - 11x + 24 \leq 0$

d $x^2 - 2x - 24 \geq 0$

e $2x^2 - 3x - 2 > 0$

f $3x^2 + 19x - 14 < 0$

g $-3 + 13x - 4x^2 \leq 0$

h $6x^2 + 16x + 8 \geq 0$

5 Complete the square or use the quadratic formula to solve these inequalities to 2 dp.

Sketch graphs to help you with these questions.

a $x^2 + 2x - 7 > 0$

b $x^2 + 7x + 8 < 0$

c $x^2 - 12x + 18 \leq 0$

d $x^2 - 3x - 21 \geq 0$

e $3x^2 - 5x - 7 > 0$

f $4x^2 + 17x - 4 < 0$

g $5x^2 - 17x + 12 \leq 0$

h $6x^2 - 16x - 7 \geq 0$

6 These inequalities define a region of the x-y plane. In each case

i Write equations which define the boundaries of the region,

ii Use algebra to find the points where the boundaries intersect,

iii Draw a graph and shade the appropriate region.

a $y < 2x + 3; y > x^2$

b $x + y \leq 4; y > x^2 - 5x + 4$

c $y - 4x \leq 17; y \leq 4x^2 - 4x - 15; x \leq 4$

d $y - 2x - 20 < 0; y + 4x - 6 < 0;$
$y > x^2 - 5x - 24$

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Reasoning and problem-solving

Strategy

To solve a problem involving inequalities

1. Use the information in the question to write the inequalities.
2. Solve the inequalites and, if requested, show them on a suitable diagram.
3. Write a clear conclusion that answers the question.

Example 4

Alan travels a journey of 200 miles in his car. He is travelling in an area with a speed limit of 70 mph.

Write down and solve an inequality in t (hours) to represent the time his journey takes.

$70 \geq \frac{200}{t}$ — (1) Use speed limit $\geq \frac{\text{distance}}{\text{time}}$

$t \geq \frac{200}{70}$

$t \geq 2.857\ldots$ — (2) Solve the inequality

$0.857\ldots \times 60 = 51.42\ldots$

The journey will take at least 2 hours 51 minutes. — (3) Write a clear conclusion.

Example 5

An illustration in a book is a rectangle $(x - 7)$ cm wide and $(x + 1)$ cm long.

It must have an area less than 65 cm^2

Work out the range of possible values of x. Justify your answer.

$(x - 7)(x + 1) < 65$ — (1) Write the inequality

$x^2 - 6x - 72 < 0$

$(x - 12)(x + 6) < 0$

$-6 < x < 12$ — (2) Solve the inequality

However, since any side of a rectangle must be positive, it follows that $x - 7 > 0$, so $x > 7$

So the solution is $7 < x < 12$ — (3) x is both greater than 7 and smaller than 12 so you combine the two inequalities.

Exercise 1.7B Reasoning and problem-solving

1 a Children in a nursery range from six months old to 4 years and six months old inclusive.

Represent this information on a number line.

b The range of temperatures outside the Met Office over a 24 hour period ranged from -4°C to 16°C.

Represent this information on a number line.

2 On a youth athletics club trip there must be at least one trainer for every six athletes and the trip is not viable unless at least eight athletes travel. Due to illness there are fewer than six trainers available to travel. Represent this information as a shaded area on a graph.

3 In an exam, students take a written paper (marked out of 100) and a practical paper (marked out of 25).

The total mark, T, awarded is gained by adding together twice the written mark and three times the practical mark. To pass the exam, T must be at least 200. A student scores w marks in the written paper and p in the practical.

a Write an inequality in w and p

b Solve this inequality for

i $w = 74$

ii $p = 9$

c Can a student pass if she misses the practical exam?

4 The length of a rectangle, $(5m + 7)$ cm, is greater than its width, $(2m + 16)$ cm

What values can m take?

5 For Amanda's 18th birthday party, 110 family and friends have been invited and at most 10% will not be able to come. Food has been prepared for 105 people.

Write down inequalities for the number of people, n, who come to the party and have enough to eat.

Solve them and find all possible solutions.

6 A bag contains green and red discs. There are r red discs and three more green than red. The total number of discs is not more than twenty. Write appropriate inequalities and find all solutions.

7 A girl is five years older than her brother. The product of their ages is greater than 50. What ages could the sister be?

8 The length of a rectangle, $(5b - 1)$ cm, is greater than its width, $(2b + 9)$ cm. The area is less than 456 cm^2. Find the possible values of b

9 The sum, S, of the first n positive integers is given by the formula $2S = n(n + 1)$. What are the possible values of n for values of S between 21 and 820?

10 The ages of two children sum to 10 and the product of their ages is greater than 16. Find all possible values of the children's ages.

Challenge

11 A firm makes crystal decanters.

The profit, £P, earned on x thousand decanters is given by the formula $P = -20x^2 + 1200x - 2500$

a Solve the equation $-20x^2 + 1200x - 2500 = 0$ giving your answer to two decimal places.

b Sketch the graph of $y = -20x^2 + 1200x - 2500$

c Use your graph to estimate

i The values of x where the firm makes a loss,

ii The range of values of x for which the profit is at least £10 000. Check this algebraically.

1 Summary and review

Chapter summary

- To use direct proof, assume P is true and then use P to show that Q must be true.
- To use proof by exhaustion, show that the cases are exhaustive and then prove each case.
- To use proof by counter example, give an example that disproves the statement.
- $x^a \times x^b = x^{a+b}$, $x^a \div x^b = x^{a-b}$, $(x^a)^b = x^{ab}$
- $x^0 = 1$, $x^{-n} = \frac{1}{x^n}$, $x^{\frac{1}{n}} = \sqrt[n]{x}$, $x^{\frac{p}{r}} = \sqrt[r]{(x^p)}$ or $(\sqrt[r]{x})^p$
- You can write any rational number exactly in the form $\frac{p}{q}$, where p and q are integers.
- $\sqrt{A} \times \sqrt{B} = \sqrt{AB}$; $\frac{\sqrt{A}}{\sqrt{B}} = \sqrt{\frac{A}{B}}$
- You rationalise a fraction in the form $\frac{k}{\sqrt{a}}$ by multiplying top and bottom by $\sqrt{a}$
- You rationalise a fraction in the form $\frac{k}{a \pm \sqrt{b}}$ by multiplying top and bottom by $a \mp \sqrt{b}$
- You rationalise a fraction in the form $\frac{k}{\sqrt{a} \pm \sqrt{b}}$ by multiplying top and bottom by $\sqrt{a} \mp \sqrt{b}$
- Any function of x in the form $ax^2 + bx + c$ where $a \neq 0$ is called a quadratic function and $ax^2 + bx + c = 0$ is called a quadratic equation.
- You can solve a quadratic equation $ax^2 + bx + c = 0$ using a calculator, by factorisation, by completing the square, by using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, and graphically.
- If the discriminant $\Delta = b^2 - 4ac > 0$, the quadratic has two different roots. If $\Delta = b^2 - 4ac = 0$, the quadratic has one repeated root. If $\Delta = b^2 - 4ac < 0$, the quadratic has no real roots.
- You can use gradients of two straight lines to decide if they are parallel, perpendicular, or neither.
- The equation of a circle, centre (a, b) and radius r, is $(x - a)^2 + (y - b)^2 = r^2$
- If you multiply or divide an inequality by a negative number you reverse the inequality sign.

Check and review

You should now be able to...	Try Questions
✔ Use direct proof, proof by exhaustion and counter examples to prove results.	1, 2, 3
✔ Use and manipulate the index laws for all powers.	4
✔ Manipulate surds and rationalise a denominator.	5, 6
✔ Solve quadratic equations using a variety of methods.	7, 8
✔ Understand and use the coordinate geometry of the straight line and of the circle.	9, 10
✔ Understand and solve simultaneous equations involving only linear or a mix of linear and non-linear equations.	11, 12
✔ Solve linear and quadratic inequalities algebraically and graphically.	13, 14

1 Prove that the product of two odd numbers must be odd.

2 Prove that there is at least one prime number between the numbers 40 and 48

3 Is it true that for every number n, $\frac{1}{n} < n$? Give a reason for your answer.

4 Simplify

a $(-s^4)^3$ b $\sqrt{64c^{64}}$ c 3^{-4} d $(k^2)^{\frac{-3}{4}}$

5 a Express $\sqrt{275}$ in its simplest form.

b Rationalise the denominator of $\frac{3-\sqrt{a}}{\sqrt{a}+1}$

6 What is the length of the hypotenuse of a right-angled triangle with sides containing the right angle of length $3\sqrt{3}$ and $3\sqrt{5}$ cm?

7 a Solve the equation $2c^2+9c-5=0$ by factorisation.

b Solve, to 2 dp, the equation $5x^2+9x-28=0$

8 Sketch the quadratic curve $y=x^2-4x-1$

9 a Write down the equations of these lines.

i Gradient -6 passing through $(6,-7)$

ii Gradient $\frac{2}{3}$ passing through $(-3,4)$

b A square joins the points $(-2,1)$, $(2,4)$, $(5,0)$ and $(1,-3)$. Write the equations of its diagonals. Hence prove that they are perpendicular.

10 a Write the equations of the circles. The centre and radius are given for each.

i $(3,6)$; 8 ii $(-3,9)$; 4 iii $(-2,-7)$; 11

b Write the equation of the tangent, at point $P(-9,19)$, to the circle with centre $(-4,7)$ and radius 13

c A circle with centre $C(5,6)$ and radius 10 has $M(8,5)$ as the midpoint of a chord. Work out the coordinates of the ends of the chord.

11 Solve these equations simultaneously.

a $2x-5y=11$; $4x+3y=9$

b $2x-3y=5$; $x^2-y^2+5=0$

12 The line $y=3x+4$ intersects the curve $xy=84$ at two points. Work out their coordinates.

13 a Writing your answers in set notation, solve these inequalities.

i $12-3x \geq 7x+2$

ii $2(x-7)+5(6-3x) \leq 10$

b Solve these inequalities, giving your answers to 2dp.

i $x^2-14x+16 \leq 0$

ii $5x^2-13x-11 \geq 0$

14 Shade the regions represented by these inequalities.

a $2y-2x-7 \leq 0$; $x+y-7<0$; $y>0$

b $x+y \leq 8$; $y>(x-2)^2-4$

What next?

Score		
	0–7	Your knowledge of this topic is still developing. To improve, search in MyMaths for the codes: 2001–2005, 2008, 2009, 2014–2018, 2020, 2021, 2025, 2026, 2033–2037, 2252, 2253, 2255–2257
	8–10	You're gaining a secure knowledge of this topic. To improve, look at the InvisiPen videos for Fluency and skills (01A)
	11–14	You've mastered these skills. Well done, you're ready to progress! To develop your techniques, look at the InvisiPen videos for Reasoning and problem-solving (01B)

Click these links in the digital book

Exploration

Going beyond the exams

History

Pierre de Fermat was a lawyer in 17th century France who studied mathematics as a hobby. He often wrote comments in the margins of the maths books that he read and, on one occasion, wrote about a problem set over a thousand years ago by Greek mathematician **Diophantus**.

The problem was to find solutions to the equation $x^n + y^n = z^n$ where x, y, z and n are all positive integers. Fermat wrote that he had discovered 'the most remarkable proof' that the equation has no solutions if $n \geq 3$, but that the margin was too small to contain it.

Despite many attempts, a copy of Fermat's proof was never found. No one else was able to prove or disprove it for over 350 years.

Have a go

Find the flaw in the following 'proof'.

$x = 1$	
$x^2 = 1$	Square both sides
$x^2 - 1 = 0$	Subtract 1 from both sides
$(x+1)(x-1) = 0$	Factorise
$x + 1 = 0$	Divide both sides by $(x - 1)$
$2 = 0$	Substitute $x = 1$

Research

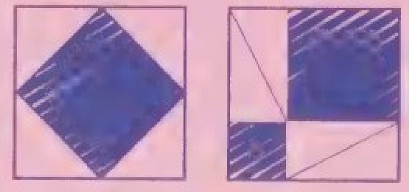
Who eventually proved the result known as **Fermat's last Theorem**?

How long did it take them to complete the proof?

Investigation

How does Fermat's last theorem relate to **Pythagoras' Theorem**?

How can you use these diagrams to prove Pythagoras' Theorem?



What are Pythagorean triples? How many are there?

"We cannot solve our problems with the same level of thinking that created them."
- Einstein

1 Assessment

1 Simplify $(3+2\sqrt{2})(5-4\sqrt{2})$. Choose the correct answer.

A $31-2\sqrt{2}$ **B** $-1+2\sqrt{2}$ **C** $-1-2\sqrt{2}$ **D** $15-10\sqrt{2}$ **[1 mark]**

2 What is the equation of the straight line that is perpendicular to $3x+2y=5$ and that passes through the point (4, 5)? Choose the correct answer.

A $3y-2x=7$ **B** $3y+2x=7$ **C** $2y-3x=7$ **D** $2y+3x=7$ **[1]**

3 **a** Simplify these expressions.

i $2^m \times 2^n$ ii $\frac{5^{m+1}}{5^{2n}}$ iii $(3^m)^2 \times \sqrt{(3^m)}$ **[5]**

b Given $\frac{16^p \times 8^q}{4^{p+q}} = 2^n$, write down the an expression for n in terms of p and q **[3]**

4 **a** Simplify these surds. You must show your working.

i $\frac{12}{\sqrt{3}}$ ii $\frac{3-\sqrt{7}}{1+3\sqrt{7}}$ **[6]**

b A rectangle $ABCD$ has an area of $8\,\text{cm}^2$ and length $(3-\sqrt{5})$ cm.

Work out its width, giving your answer as a surd in simplified form. Show your working. **[3]**

5 **a** Express $x^2+6x+13$ in the form $(x+a)^2+b$ **[2]**

b Hence sketch the curve $y=x^2+6x+13$ and label the vertex, and the point where the curve cuts the y-axis. **[3]**

6 Solve these simultaneous equations.

$2x+y=3$ $3x^2+2xy+7=0$ **[8]**

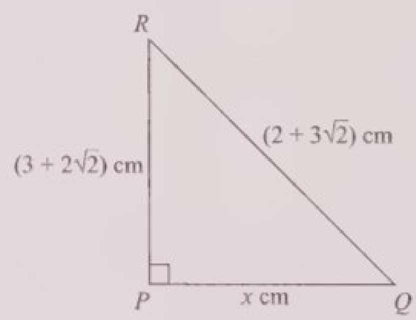
7 Prove that the equation $x=1+\frac{2x-5}{x+4}$ has no real solutions. **[4]**

8 **a** Solve these inequalities.

i $3x-5<11-x$ ii $x^2-6x+5\le 0$ **[5]**

b Show on a graph the set of values of x that satisfy both $3x-5<11-x$ and $x^2-6x+5\le 0$ **[2]**

9 PQR is a right-angled triangle.
Write an exact expression for x, show your working. **[6]**



10 The equation of a circle is $x^2+y^2-10x+2y-23=0$

a Work out **i** Its centre, **ii** Its radius. **[5]**

b The line $y=x+2$ meets the circle at the points P and Q. Work out, in exact form, the coordinates of P and Q **[5]**

11 The quadratic equation $(k+1)x^2-4kx+9=0$ has distinct real roots.
What range of values can k take? **[6]**

12 Prove that $\frac{a+b}{2}\ge\sqrt{ab}$ for all positive numbers a and b **[4]**

13 **a** Factorise the expression $2u^2 - 17u + 8$ **[2]**

b Hence solve the equation $2^{2x+1} - 17 \times 2^x + 8 = 0$ **[3]**

14 The straight line $y = mx + 2$ meets the circle $x^2 + y^2 + 4x - 6y + 10 = 0$

a Prove that the x-values of the points of intersection satisfy the equation

$(m^2 + 1)x^2 + 2(2 - m)x + 2 = 0$ **[4]**

b The straight line $y = mx + 2$ is a tangent to the circle $x^2 + y^2 + 4x - 6y + 10 = 0$

What are the possible values of m? Give your answers in exact form. **[5]**

15 **a** Given $9^2 = 3^n$, write down the value of n **[1]**

b Solve these simultaneous equations.

$3^{x+y} = 9^2$ $\quad\quad$ $4^{x-2y} = 8^4$ **[4]**

16 Decide which of these statements are true and which are false.
For those that are true, prove that they are true.
For those that are false, give a counter-example to show that they are false.

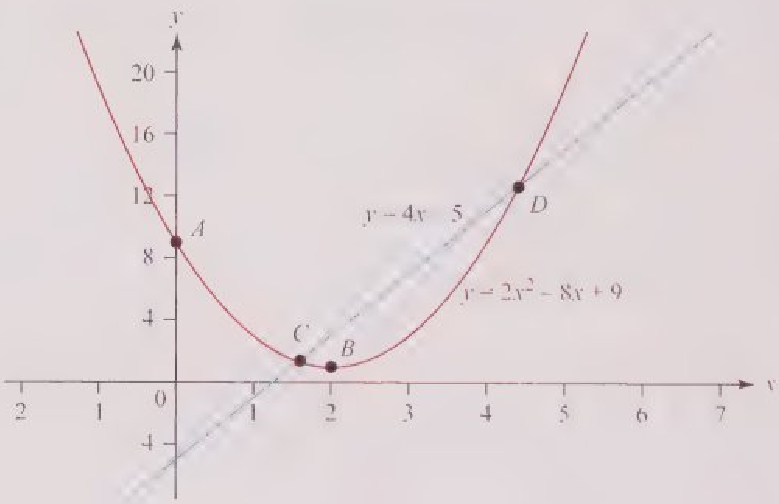
a If $a > b$ then $a^2 > b^2$ **[2]**

b $n^2 + n$ is an even number for all positive integers n **[3]**

c If a and b are real numbers then $b^2 \geq 4a(b - a)$ **[3]**

d $2^n - 1$ is prime for all positive integers n **[2]**

17 The diagram shows the parabola $y = 2x^2 - 8x + 9$ and the straight line $y = 4x - 5$



Work out the coordinates of the following points.

a A, the y-intercept of the parabola. **[1]**

b B, the vertex of the parabola. **[2]**

c C and D, the points of intersection of the line and parabola. **[4]**

18 Prove that the circle $x^2 + y^2 + 6x - 4y - 2 = 0$
lies completely inside the circle $x^2 + y^2 - 2x - 10y - 55 = 0$ **[9]**